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A note on the path integral representation for Majorana fermions

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Abstract

Majorana fermions are currently of huge interest in the context of nanoscience and condensed matter physics. Different to usual fermions, Majorana fermions have the property that the particle is its own anti-particle thus, they must be described by real fields. Mathematically, this property makes nontrivial the quantization of the problem due, for instance, to the absence of a Wick-like theorem. In view of the present interest on the subject, it is important to develop different theoretical approaches in order to study problems where Majorana fermions are involved. In this note we show that Majorana fermions can be studied in the context of field theories for constrained systems. Using the Faddeev–Jackiw formalism for quantum field theories with constraints, we derived the path integral representation for Majorana fermions. In order to show the validity of the path integral we apply it to an exactly solvable problem. This application also shows that it is rather simple to perform systematic calculations on the basis of the present framework.

Keywords: Majorana fermions, Faddeev–Jackiw quantization, path integral

1. Introduction

In 1937 Ettore Majorana discovered the real solution of the Dirac equation [1], and he also speculated that his solution might describe the physics of the neutrinos. Since that time, Majorana fermions are a matter of intense studies in particle physics [2]. However, in the last years great interest on them occurred in condensed matter physics [3, 4]. For instance, the pioneer works from Kitaev [5] have pushed the topic towards topological phases in spin systems (see [6] and references therein). The recent observation of Majorana fermions in ferromagnetic chains on a superconductor is also very remarkable [7]. In this context it is worth to note that in the period 2014–2015 about 500 papers containing the word Majorana in the abstract were reported only in the condensed-matter-section of arXiv.

Majorana fermions have the property that the particle is its own anti-particle. This property leads to commutation rules which differ considerably from the commutation rules for usual fermions. As it is well known, the commutation rules for fermions are a key point for the validity of the Wick theorem, and the consequent theoretical quantum treatment of many-body systems. Thus, the canonical quantization of Majorana fermions is not trivial. An alternative theoretical framework for quantization is via the path integral representation. Previous work [8] focused on this problem adapting the coherent state formalism [9] to Majorana fermions. Since there are not coherent states for Majorana fermions the derivation can be done using pairs of conventional fermions. Thus, the path integral formulation is not straightforward (see also [10]). The fact that Majorana fermions is a timely subject it is important to develop a new alternative method for such derivation.

In this note we show that a theory containing Majorana fermions can be considered as a constrained system. The treatment of constrained systems was initiated by Dirac [11] and continued by Faddeev and Jackiw (FJ) [12, 13]. This approach, which is well known in field theory, allows to write the corresponding path integral for the model after obtaining the classical effective action consistent with the corresponding quantum theory. Thus, the path integral allows to study the quantum problem and for instance read the Feynman rules from the effective action [14]. The FJ method was used also in solid state physics for the Heisenberg model [15], $t - J$ model [16], and for bond-fields in spin systems [17].

The paper is organized as follows. In section 2 we present the FJ method for Majorana fermions and derive the corresponding path integral representation. In section 3 we show the validity of the path integral by applying it to a simple and exactly solvable problem. In addition, this example shows how to work systematically with this formulation. Discussion and conclusion are given in section 4.

2. Path integral formulation for Majorana fermions

2.1. Quantum algebra for Majorana fermions: difficulties about the canonical quantization

Usual fermions are described by the well known creation and destruction operators $\hat{f}_a^\dagger$ and $\hat{f}_a$, respectively, that satisfy the following anticommutation rules

$$\{\hat{f}_a, \hat{f}_b^\dagger\} = \hat{f}_a \hat{f}_b^\dagger + \hat{f}_b^\dagger \hat{f}_a = \delta_{ab}. \quad (2.1)$$

Mathematically, the property that for Majorana fermions the particle is its own anti-particle is expressed as $\hat{f}_a^\dagger = \hat{f}_a$. Thus, the commutation rules for Majorana fermions become

$$\{\hat{f}_a, \hat{f}_b^\dagger\} = \{\hat{f}_a^\dagger, \hat{f}_b\} = \{\hat{f}_a^\dagger, \hat{f}_b^\dagger\} = \{\hat{f}_a, \hat{f}_b\} = \delta_{ab}, \quad (2.2)$$

leading, for instance, to the unusual property $\hat{f}_a \hat{f}_a = 1/2$.

The commutation rules (2.2) makes the quantum problem complicated. At the operatorial or canonical level one way is to work in the Hilbert space generated by $\hat{f}_a^\dagger$ and $\hat{f}_a$, and the constraint between operators $\hat{\Omega} = \hat{f}_a^\dagger - \hat{f}_a = 0$ should be applied rigorously. At this point it is worth to mention that the commutation rules for Majorana fermions (2.2) make difficult the formulation of a Wick theorem and thus, the quantum many-body treatment is not trivial. Instead of using the canonical quantization, in the next subsection we introduce the path integral representation for Majorana fermions.

2.2. FJ theory and path integral for Majorana fermions

In order to formulate a path integral representation for quantum systems, the coherent-state approach [9] is often used. Instead of that, here we use the formalism introduced by FJ [12, 13, 18] for constrained system, that provides a way for obtaining a classical theory consistent with the algebra of the quantum problem. In this approach, no formal distinction is made between different forms of constraints like in Dirac's theory [11], where primary and secondary, first class and second class constraints appear. In the FJ method [12, 13, 18] the constraints can be incorporated iteratively until the basic brackets for the fields can be determined. Once the classical fieldtheory is obtained, quantization can proceed via a path integral or in a canonical way. It is worth to remember that a path integral formulation is written in terms of classical variables, Grassmann or complex regular numbers for fermionic or bosonic fields [19], respectively. Since the FJ-formalism is not widely used, we will briefly discuss it.

Following the FJ theory [12] and its extension to fermionic degrees of freedom [13], we defined the following effective first-order classical Lagrangian

$$L = \sum_A \dot{z}_A K^A(z_A) - H(z_A), \quad (2.3)$$

where z_A are fermionic or bosonic fields, $H(z_A)$ is the Hamiltonian, and the coefficients $K^A(z_A)$ are functions of the fields z_A (see below).

It is possible to prove [12] that if the matrix

$$M_{AB} = \frac{\partial K^B}{\partial z_A} - (-1)^{\epsilon_A \epsilon_B} \frac{\partial K^A}{\partial z_B} \quad (2.4)$$

is not singular, the theory is unconstrained. In (2.4) $\epsilon_A = 1, 0$ if z_A is a fermionic or bosonic field, respectively. In this case the basic bracket (or generalized bracket) of the FJ theory, defined as

$$\{z_A, z_B\}_{\text{FJ}} = (-1)^{\epsilon_A} M_{AB}^{-1}, \quad (2.5)$$

agrees with the Poisson bracket. M_{AB}^{-1} is the element (AB) of the inverse of the matrix M_{AB} .

If the theory contains a constraint $\Omega(z_A) = 0$, this can be included using a Lagrange multiplier ξ ,

$$L = \sum_A \dot{z}_A K^A(z_A) - H(z_A) + \xi \Omega(z_A). \quad (2.6)$$

In this case the matrix M_{AB} is singular and its zero mode encodes information of the constraint.

Following [12], the variable ξ disappears after imposing the constraint, leading to the first-iterated Lagrangian defined as

$$L = \sum_A \dot{z}_A K^A(z_A) + \dot{\lambda} \Omega(z_A) - H(z_A). \quad (2.7)$$

In (2.7) the space was enlarged as $z_A \longrightarrow (z_A, \lambda)$, and $K^A(z_A) \longrightarrow (K^A(z_A), \Omega(z_A))$.

Now, the new matrix M_{AB} is nonsingular and the FJ generalized bracket (2.5) agrees with the Dirac bracket [11] of the constrained theory.

As usual, to pass from the classical theory to the quantum theory we have to associate each classical variable with the corresponding operator, and impose the commutation between them. In the FJ and Dirac theories the prescription is

$$[\hat{z}_A, \hat{z}_B]_{\pm} = -i\{z_A, z_B\}_{\text{FJ}}, \quad (2.8)$$

where $+$ ($-$) is for an anticommutator (commutator).

Note that we use the hat symbol for quantum operators, while the variables without that symbol are classical variables.

It is important to note that (2.5)–(2.8) may be used to obtain the commutation rules of the quantum theory when the coefficients $K^A(z_A)$ are known. On the other hand, if the commutation rules are known, as in the case of Majorana fermions, (2.5)–(2.8) can be used for deriving the explicit expression for the coefficients $K^A(z_A)$ introduced in (2.3).

Next, we will review the above ideas applying the FJ formalism to our problem of interest. Following (2.7) we propose the first-iterated Lagrangian for the Majorana fermions

$$L = \sum_a [\dot{f}_a^\dagger A_a^{f^\dagger}(f, f^\dagger) + \dot{f}_a A_a^f(f^\dagger, f) + \dot{\lambda}_a \Omega_a] - H(f^\dagger, f), \quad (2.9)$$

where f_a and $f_a^\dagger$ are complex fermionic Grassmann variables, and λ is a Grassmann Lagrangian multiplier associated with the constraint $\Omega_a = f_a^\dagger - f_a = 0$. In (2.9) H is a proper model Hamiltonian.

Note that at the classical level f_a and $f_a^\dagger$ are considered as Grassmann fields, and for instance $f_a f_a = 0$ due to the properties of the Grassman algebra [19], which should not be confused with an operatorial rule.

Identifying the set of variables $(f_a^\dagger, f_a, \lambda_a)$, and the coefficients $(A_a^{f^\dagger}(f, f^\dagger), A_a^f(f^\dagger, f), \Omega_a)$, the explicit expression for the unknown coefficients $A_a^{f^\dagger}(f, f^\dagger)$ and $A_a^f(f^\dagger, f)$ in (2.9) can be determined in such a way that the matrix M_{AB} leads to the commutation rules (2.2) for Majorana fermions.

After some algebra we obtain

$$A_a^{f^\dagger}(f, f^\dagger) = \frac{i}{4} f_a, \quad (2.10)$$

and

$$A_a^f(f, f^\dagger) = \frac{i}{4} f_a^\dagger. \quad (2.11)$$

Thus, the explicit expression for L is

$$L = \sum_a \left[\frac{i}{4} \dot{f}_a^\dagger f_a + \frac{i}{4} \dot{f}_a f_a^\dagger + \dot{\lambda}_a \Omega_a \right] - H(f^\dagger, f). \quad (2.12)$$

It is useful to show that the commutation rules for Majorana fermions are recovered. Using (2.4) M_{AB} can be easily calculated, and its inverse is

$$M_{AB}^{-1} = \begin{pmatrix} i & i & -\frac{1}{2} \\ i & i & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{i}{4} \end{pmatrix} \delta_{ab}. \quad (2.13)$$

Thus, the FJ brackets between f_a and $f_b^\dagger$ are

$$\{f_a, f_b^\dagger\}_{\text{FJ}} = \{f_a^\dagger, f_b\}_{\text{FJ}} = \{f_a^\dagger, f_b^\dagger\}_{\text{FJ}} = \{f_a, f_b\}_{\text{FJ}} = i\delta_{ab}. \quad (2.14)$$

Therefore, using (2.8) the commutation rules for Majorana fermions are obtained.

Once the classical effective Lagrangian (2.12) that leads to the proper quantum algebra (2.2) is obtained, the next step is to write the path integral. Following [20, 21]

$$\int \mathcal{D}f \mathcal{D}f^\dagger (\det M_{AB})^{1/2} \delta(\Omega) e^{i \int dt L'}, \quad (2.15)$$

where $L' = L|_{\Omega=0}$, i.e.

$$L' = \sum_a \left[\frac{i}{4} \dot{f}_a^\dagger f_a + \frac{i}{4} \dot{f}_a f_a^\dagger \right] - H(f^\dagger, f), \quad (2.16)$$

and $\det M_{AB}$ is the determinant of M_{AB} .

In contrast to other theories [16, 17] $\det M_{AB}$ does not depend on the fields and can be absorbed into the measure of the path integral. The integration on $f_a^\dagger$ can be done using the $\delta(\Omega)$. Thus, (2.15) is

$$\int \mathcal{D}f e^{i \int dt L_{\text{eff}}} \quad (2.17)$$

where

$$L_{\text{eff}} = \sum_a \frac{i}{2} \dot{f}_a f_a - H(f). \quad (2.18)$$

At this point we make two formal remarks: (a) the kinetic action for the case of Majorana fermions contains a factor 1/4 (2.12) instead of 1/2 as for usual fermions [22]. One may intuitively think that the effective Lagrangian for Majorana fermions is that of the usual fermions where the kinetic action contains a factor 1/2, plus the constraint $\Omega = 0$ for imposing the reality of the fields. However, it is possible to show that this proposal does not lead to the correct commutation rules (2.2). (b) The kinetic term of (2.18) is $\frac{i}{2} \dot{f}f$. For the case of a real boson b , a kinetic term of the form $\dot{b}b$ can not exist, because it can be expressed as a total derivative respect to time, $\dot{b}b = \frac{1}{2} \frac{\partial (bb)}{\partial t}$. Thus, $\dot{b}b$ can be neglected in the effective action. However, for Grassmann variables $\frac{\partial (ff)}{\partial t} = \dot{f}f - f\dot{f} = 0$, which means that $\dot{f}f$ can not be written as a total derivative with respect to time and can not be neglected.

3. The Majorana propagator and an example of application

3.1. An exactly solvable problem

In this section we apply the path integral representation obtained in the last section to a simple and exactly solvable problem. There are two aims for this calculation: (a) to show that the path integral is well defined and at the same time (b) to show how systematic calculations can be performed on this framework.

As in [8] we propose the following Hamiltonian

$$\hat{H} = \epsilon (\hat{c}^\dagger \hat{c} - 1/2), \quad (3.1)$$

where $\hat{c}^\dagger$ ($\hat{c}$) creates (annihilates) a fermion in an orbital of energy ϵ . The energy was trivially shifted by $-\epsilon/2$ for convenience. Using the standard many-body theory it is straightforward to show that $\langle H \rangle = -\frac{\epsilon}{2} \tanh(\frac{\epsilon}{2k_B T})$, where T is the temperature.

It is easy to see that using the commutation rules for Majorana fermions, $\hat{c}$ can be written as $\hat{c} = (\hat{\chi}_1 - i\hat{\chi}_2)/\sqrt{2}$, where $\hat{\chi}_1$ and $\hat{\chi}_2$ are two Majorana fermions.

In terms of $\hat{\chi}_1$ and $\hat{\chi}_2$ the Hamiltonian is:

$$\hat{H} = -i\epsilon \hat{\chi}_1 \hat{\chi}_2. \quad (3.2)$$

Following the results of the previous section we first remove the hat symbol and introduce the classical Hamiltonian

$$H = -i\epsilon\chi_1\chi_2, \quad (3.3)$$

where now χ_1 and χ_2 are two Grassmann variables.

Second, following (2.17), the path integral for this problem reads

$$\int \mathcal{D}\chi_1 \mathcal{D}\chi_2 e^{i \int dt L_{\text{eff}}}, \quad (3.4)$$

where the classical effective Lagrangian is

$$L_{\text{eff}} = \frac{i}{2}(\dot{\chi}_1 \chi_1 + \dot{\chi}_2 \chi_2) + i\epsilon\chi_1\chi_2. \quad (3.5)$$

Going to the Euclidean time $\tau = it$, the path integral partition function, which allows the many-body calculation at finite temperature [23], is

$$Z = \int \mathcal{D}\chi_1 \mathcal{D}\chi_2 e^{-S_{\text{eff}}}, \quad (3.6)$$

where

$$S_{\text{eff}} = \int d\tau L_{\text{eff}} = \int d\tau \left[\frac{1}{2}(\dot{\chi}_1 \chi_1 + \dot{\chi}_2 \chi_2) - i\epsilon\chi_1\chi_2 \right]. \quad (3.7)$$

Performing the Fourier transformation

$$\chi_i(\tau) = \sum_{i\omega_n} e^{i\omega_n \tau} \chi_i(i\omega_n), \quad (3.8)$$

where $i\omega_n$ are fermionic Matsubara frequencies, L_{eff} can be written as

$$L_{\text{eff}} = \frac{1}{2} \sum_{i\omega_n} \sum_{i,j} \chi_i(i\omega_n) D_{ij}^{-1}(i\omega_n) \chi_j(-i\omega_n), \quad (3.9)$$

where

$$D_{ij}^{-1} = \begin{pmatrix} i\omega_n & -i\epsilon \\ i\epsilon & i\omega_n \end{pmatrix}, \quad (3.10)$$

and i and j take values 1 and 2.

Once the effective Lagrangian is defined, the Feynman rules can be read from L_{eff} [14]: the inverse of the quadratic parts in the fields define the propagator and the remaining terms the interaction vertices.

From (3.9) and (3.10) the 2×2 fermionic propagator $D_{ij}(i\omega_n)$ is:

$$D_{ij}(i\omega_n) = \frac{1}{(i\omega_n - \epsilon)(i\omega_n + \epsilon)} \begin{pmatrix} i\omega_n & i\epsilon \\ -i\epsilon & i\omega_n \end{pmatrix}, \quad (3.11)$$

Note that due to the real nature of the variables χ 's the propagator must be defined without the factor 1/2 in (3.9).

$D_{ij}(i\omega_n)$ must be associated with the propagator $\langle T(\hat{\chi}_i(\tau) \hat{\chi}_j(0)) \rangle$ for the Majorana fermions in the $i\omega_n$ space. The average $\langle \chi_1 \chi_2 \rangle$ needed for the calculation of $\langle H \rangle$ is

$$\langle \chi_1 \chi_2 \rangle = \langle T \chi_1(\tau = 0^+) \chi_2(0) \rangle = \sum_{i\omega_n} e^{i\omega_n 0^+} D_{12}(i\omega_n). \quad (3.12)$$

After performing the sum over the fermionic Matsubara frequencies we obtain

$$\langle \chi_1 \chi_2 \rangle = \sum_{i\omega_n} \frac{i\epsilon}{(i\omega_n - \epsilon)(i\omega_n + \epsilon)} = -\frac{i}{2} \tanh\left(\frac{\epsilon}{2k_B T}\right). \quad (3.13)$$

Thus, (3.13) leads to the correct result for $\langle H \rangle$.

3.2. Some characteristics of the Majorana propagator

From (3.11) we can also reconstruct the usual fermionic propagator $G(\tau) = \langle T\hat{c}(\tau)\hat{c}^\dagger(0) \rangle$ [22]. From the relation between $\hat{c}$ and the $\hat{\chi}$'s we have

$$G(\tau) = \langle T\hat{\chi}_1(\tau)\hat{\chi}_1(0) \rangle + \langle T\hat{\chi}_2(\tau)\hat{\chi}_2(0) \rangle + i\langle T\hat{\chi}_1(\tau)\hat{\chi}_2(0) \rangle - i\langle T\hat{\chi}_2(\tau)\hat{\chi}_1(0) \rangle. \quad (3.14)$$

In the Matsubara space, and using the notation of (3.11), we get

$$G(i\omega_n) = D_{11}(i\omega_n) + D_{22}(i\omega_n) + D_{12}(i\omega_n) + D_{21}(i\omega_n) = \frac{1}{(i\omega_n - \epsilon)} \quad (3.15)$$

which is the propagator or the Green function for usual fermions [22].

Finally, we note that the propagators for χ_1 and χ_2 , $D_{11}(i\omega_n)$ and $D_{22}(i\omega_n)$ respectively, are odd in $i\omega_n$ as discussed in [24]. In contrast, the off-diagonal elements $D_{12}(i\omega_n)$ and $D_{21}(i\omega_n)$ are even in $i\omega_n$. These off-diagonal elements can be thought also as coming from the coupling between Majorana fermions (see also [24] for discussions). The two Majorana propagators $D_{11}(i\omega_n)$ and $D_{22}(i\omega_n)$ resonate at 0 and $-\epsilon$. These two poles are linked to the presence of the off-diagonal elements of D_{ij} , i.e., to the coupling between the Majorana fermions. Without the coupling between χ_1 and χ_2 the corresponding propagators have zero energy modes.

4. Discussion and conclusion

We have shown that models containing Majorana fermions can be identified as constrained systems. Applying the FJ method, developed originally for constrained field theories, we have derived a proper effective classical Lagrangian written in terms of Grassmann variables, which are associated with Majorana fermions at the classical level. This effective Lagrangian leads to generalized FJ brackets, the equivalent to the Poisson brackets of the classical mechanics, which at the quantum level reproduces the noncanonical commutation rules for Majorana fermions. Based on this effective classical Lagrangian, the path integral was defined. In order to show the validity of this path integral we have applied it to an exactly solvable problem, finding the correct result. Besides of that, this calculation is also useful to show how to perform systematic calculations in a straightforward way. This is important considering the present interest in this topic in the context of condensed matter physics, where different analytical and numerical methods are proposed. Finally, the method presented here may be applied to systems involving interacting Majorana fields. For instance, a Hubbard-Stratonovich transformation [22] leads to an effective theory which is quadratic in the original fermions, and these fermions interact with a bosonic decoupling field. The quadratic part in the Majorana fields, which contains information of the original interacting problem, defines a propagator of the same nature of that discussed here.

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