

Linearized stability of charged thin-shell wormholes

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Abstract

The linearized stability of charged thin shell wormholes under spherically symmetric perturbations is analyzed. It is shown that the presence of a large value of charge provides stabilization to the system, in the sense that the constraints onto the equation of state are less severe than for non-charged wormholes.

Keywords: Lorentzian wormholes; exotic matter; Einstein-Maxwell spacetimes

1 Introduction

Lorentzian wormholes were originally found as solutions of the Einstein field equations with non-trivial topology. In their more simple examples they present two mouths in asymptotically flat spacetimes connected by a throat [1]. More general and purely geometric definitions allow to remove the requirement for asymptotically flat regions and the introduction of trivial topologies in wormhole spacetimes, both static [2] and dynamic [3]. All these wormholes, nonetheless, must be threaded by matter that violates the null energy condition if they are going to be traversable by material systems [1, 2, 3].

The requirements for exotic matter that violates the energy conditions can be reduced introducing scalar fields or electric charges. Very recently Kim and Lee [4] have presented exact solutions for a static charged wormhole. These spacetimes are free of horizons, so the charge, although having effects on the stability, does not seriously affect the spacetime itself.

Despite nothing is known about the equation of state of exotic matter, a stability analysis is possible through linearized perturbations around static solutions of the Einstein field equations. This imposes constraints onto the state equation and the stability domain of the solutions. This approach has been implemented by Poisson and Visser for spherically symmetric thin-shell wormholes [5, 6]. For surgical techniques to construct thin-shell wormholes, the reader is referred to the papers by Visser [7, 8].

In this paper we extend the stability analysis to the case of charged wormholes with the aim to establish whether the presence of charge effectively can increase the stability domain of this type of

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spacetimes. We do not intend to provide any explanation on the mechanisms that might allow the wormhole to acquire or maintain its charge, but we rather focus on the consequences of the charge on the global stability of the system.

2 Charged thin shell wormholes

We use geometrized units ($c=G=1$). The Reissner-Nordström metric is the unique spherically symmetric solution of the vacuum Einstein-Maxwell coupled equations. In Schwarzschild coordinates $X^\alpha = (t, r, \theta, \varphi)$ it takes the form

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (1)$$

where M is the mass and Q^2 is the sum of the squares of the electric (Q_E) and magnetic (Q_M) charges, measured by distant observers. In a local orthonormal frame, the nonzero components of the electromagnetic field tensor are $F^{\hat{t}\hat{r}} = E^{\hat{r}} = Q_E/r^2$ and $F^{\hat{\theta}\hat{\varphi}} = B^{\hat{r}} = Q_M/r^2$. If $|Q| \leq M$, this geometry represents a black hole with an event horizon with radius

$$r_h = M + \sqrt{M^2 - Q^2}. \quad (2)$$

If $|Q| > M$ it is a naked singularity.

From the Reissner-Nordström geometry we can take two copies of the region with $r \geq a$:

$$\mathcal{M}^\pm = \{x/r \geq a\}, \quad (3)$$

where $a > r_h$ if $|Q| \leq M$ and $a > 0$ if $|Q| > M$, and glue them together at the hypersurface

$$\Sigma \equiv \Sigma^\pm = \{x/r - a = 0\}, \quad (4)$$

to make a charged spherically symmetric thin shell wormhole $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$. This construction creates a geodesically complete manifold \mathcal{M} with two asymptotically plane regions connected by a throat. On \mathcal{M} we can define a new radial coordinate $l = \pm \int_a^r g_{rr} dr$, where the positive and negative signs correspond, respectively, to \mathcal{M}^+ and \mathcal{M}^- . $|l|$ represents the proper radial distance to the throat, which is placed in $l = 0$. Distant observers in \mathcal{M} will see the wormhole as having mass M and charges Q_E and Q_M . To study this traversable wormhole we use the standard Darmois-Israel formalism [9]. For a modern treatment of this formalism see for example [10].

The wormhole throat is placed at Σ . This shell is a synchronous timelike hypersurface. We can adopt coordinates $\xi^i = (\tau, \theta, \varphi)$ in Σ , with τ the proper time on the shell. In order to analyze the stability under spherically symmetric perturbations, we let the radius of the throat be a function of the proper time, $a = a(\tau)$. Then

$$\Sigma : f(r, \tau) = r - a(\tau) = 0. \quad (5)$$

The second fundamental forms associated with the two sides of the shell are:

$$K_{ij}^\pm = -n_\gamma^\pm \left(\frac{\partial^2 X^\gamma}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^\gamma \frac{\partial X^\alpha}{\partial \xi^i} \frac{\partial X^\beta}{\partial \xi^j} \right) \Big|_\Sigma, \quad (6)$$

where n_γ^\pm are the unit normals ($n^\gamma n_\gamma = 1$) to Σ in \mathcal{M} :

$$n_\gamma^\pm = \pm \left| g^{\alpha\beta} \frac{\partial f}{\partial X^\alpha} \frac{\partial f}{\partial X^\beta} \right|^{-1/2} \frac{\partial f}{\partial X^\gamma}. \quad (7)$$

In the orthonormal basis $\{e_{\hat{\tau}}, e_{\hat{\theta}}, e_{\hat{\varphi}}\}$ ($e_{\hat{\tau}} = e_\tau$, $e_{\hat{\theta}} = r^{-1}e_\theta$, $e_{\hat{\varphi}} = (r \sin \theta)^{-1}e_\varphi$, $g_{i\hat{j}} = \eta_{i\hat{j}}$) we have that

$$K_{\hat{\theta}\hat{\theta}}^\pm = K_{\hat{\varphi}\hat{\varphi}}^\pm = \pm \frac{1}{a} \sqrt{1 - \frac{2M}{a} + \frac{Q^2}{a^2} + \dot{a}^2}, \quad (8)$$

and

$$K_{\hat{\tau}\hat{\tau}}^\pm = \mp \frac{\ddot{a} + \frac{M}{a^2} - \frac{Q^2}{a^3}}{\sqrt{1 - \frac{2M}{a} + \frac{Q^2}{a^2} + \dot{a}^2}}, \quad (9)$$

where the dot means $d/d\tau$.

The Einstein equations on the shell reduce to the Lanczos equations:

$$-[K_{i\hat{j}}] + K g_{i\hat{j}} = 8\pi S_{i\hat{j}}, \quad (10)$$

where $[K_{i\hat{j}}] \equiv K_{i\hat{j}}^+ - K_{i\hat{j}}^-$, $K = tr[K_{i\hat{j}}] = [K_{\hat{i}}^{\hat{i}}]$ and $S_{i\hat{j}} = \text{diag}(\sigma, -\vartheta_1, -\vartheta_2)$ is the surface stress-energy tensor, with σ the surface energy density and $\vartheta_{1,2}$ the surface tensions. Then replacing Eqs. (8) and (9) in Eq. (10) we obtain

$$\sigma = \frac{-1}{2\pi a} \sqrt{1 - \frac{2M}{a} + \frac{Q^2}{a^2} + \dot{a}^2}, \quad (11)$$

$$p = \frac{1}{4\pi a} \frac{1 - \frac{M}{a} + \dot{a}^2 + a\ddot{a}}{\sqrt{1 - \frac{2M}{a} + \frac{Q^2}{a^2} + \dot{a}^2}}. \quad (12)$$

where $p = -\vartheta_1 = -\vartheta_2$ is the surface pressure. The surface energy density is negative, indicating the presence of *exotic matter* in the throat.

The dynamical evolution of the wormhole throat is governed by the Einstein equations plus the equation of state $p = p(\sigma)$ that relates p and σ . Replacing the equation of state in Eq. (12) and using Eq. (11), a second order differential equation is obtained for $a(\tau)$. This equation has an unique solution for given initial conditions $a(\tau_0)$ and $\dot{a}(\tau_0)$, where τ_0 is an arbitrary (but fixed) time.

From the Eqs. (11) and (12) above, it is easy to verify the energy conservation equation:

$$\frac{d}{d\tau} (\sigma A) + p \frac{dA}{d\tau} = 0, \quad (13)$$

where $A = 4\pi a^2$ is the area of the wormhole throat. The first term of Eq. (13) represents the internal energy change of the throat and the second the work done by the internal forces of the throat. As in the case of uncharged wormholes [6], the conservation equation can be written in the form

$$\dot{\sigma} = -2(\sigma + p) \frac{\dot{a}}{a}, \quad (14)$$

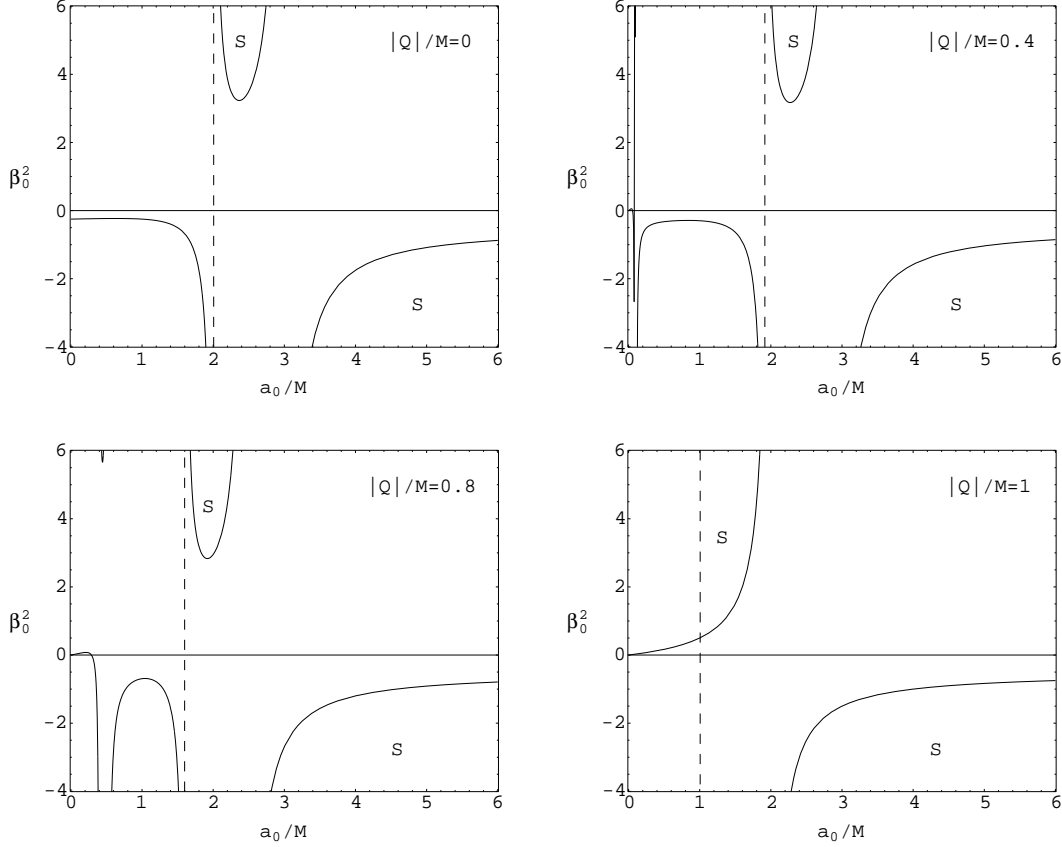


Figure 1: Stability regions (S) for different values of charge with $|Q|/M \leq 1$. The zones situated at the left of the dashed vertical lines have no physical meaning (they correspond to $a_0 \leq r_h$).

which can be integrated to give

$$\ln \frac{a}{a(\tau_0)} = -\frac{1}{2} \int_{\sigma(\tau_0)}^{\sigma} \frac{d\sigma}{\sigma + p(\sigma)}. \quad (15)$$

This can be formally inverted to obtain $\sigma = \sigma(a)$. Thus, replacing $\sigma(a)$ in Eq. (11) and regrouping terms, the dynamics of the wormhole throat is completely determined by a single equation:

$$\dot{a}^2 - \frac{2M}{a} + \frac{Q^2}{a^2} - 4\pi^2 a^2 [\sigma(a)]^2 = -1, \quad (16)$$

with the initial conditions $a(\tau_0)$ and $\dot{a}(\tau_0)$.

3 Stability

Eq. (16) can be rewritten in the form:

$$\dot{a}^2 = -V(a), \quad (17)$$

with

$$V(a) = 1 - \frac{2M}{a} + \frac{Q^2}{a^2} - 4\pi^2 a^2 [\sigma(a)]^2. \quad (18)$$

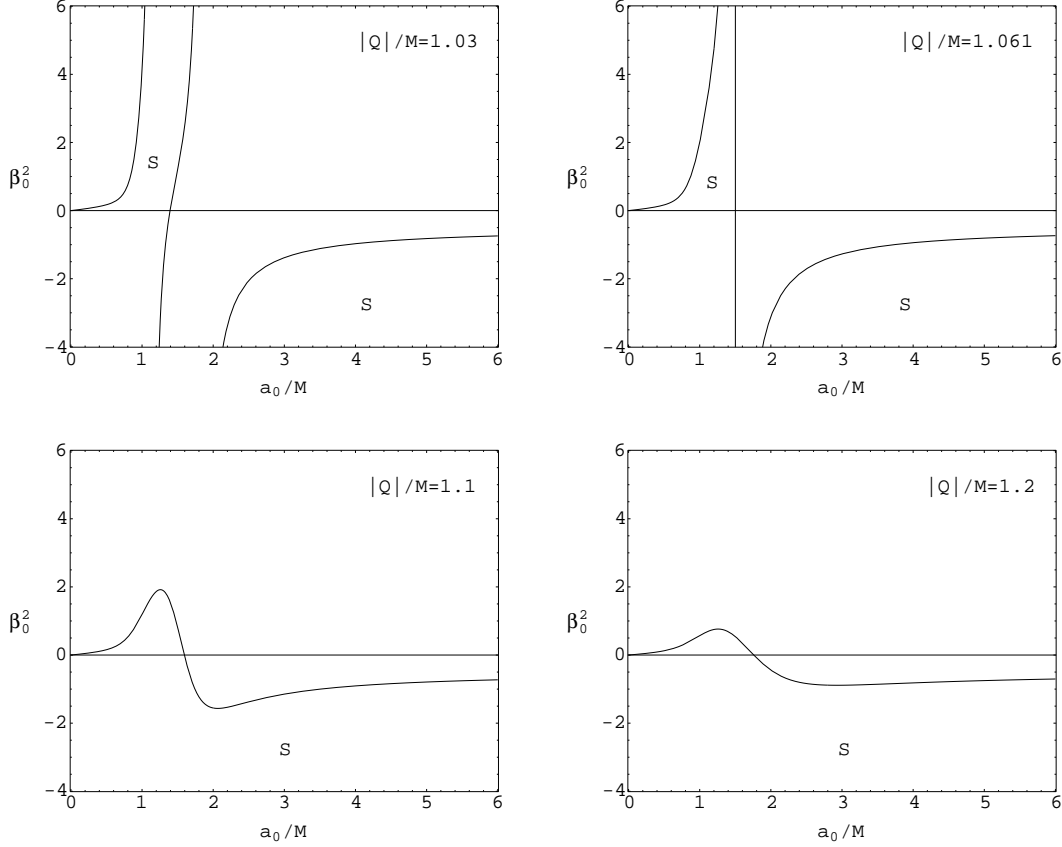


Figure 2: Stability regions (S) for different values of charge with $|Q|/M > 1$.

Let us consider now a static solution with radius a_0 and the corresponding values of energy density σ_0 and pressure p_0 :

$$\sigma_0 = \frac{-1}{2\pi a_0} \sqrt{1 - \frac{2M}{a_0} + \frac{Q^2}{a_0^2}}, \quad (19)$$

$$p_0 = \frac{1}{4\pi a_0} \frac{1 - \frac{M}{a_0}}{\sqrt{1 - \frac{2M}{a_0} + \frac{Q^2}{a_0^2}}}. \quad (20)$$

To study the stability of this static solution under radial perturbations, we carry out a linearized analysis, as it is done for the noncharged case in [6]. A Taylor expansion to second order of the potential $V(a)$ around the static solution yields:

$$V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{V''(a_0)}{2}(a - a_0)^2 + O(a - a_0)^3, \quad (21)$$

where the prime means $\frac{d}{da}$. From Eq. (18) the first derivative of $V(a)$ is

$$V'(a) = \frac{2M}{a^2} - \frac{2Q^2}{a^3} - 8\pi^2 a \sigma (\sigma + a\sigma'). \quad (22)$$

Since $\sigma' = \dot{\sigma}/\dot{a}$, using the Eq. (14), we have that $\sigma' = -2(\sigma + p)/a$. Then

$$V'(a) = \frac{2M}{a^2} - \frac{2Q^2}{a^3} + 8\pi^2 a \sigma (\sigma + 2p). \quad (23)$$

The second derivative of the potential is

$$V''(a) = -\frac{4M}{a^3} + \frac{6Q^2}{a^4} + 8\pi^2 [(\sigma + a\sigma')(\sigma + 2p) + a\sigma(\sigma' + 2p')]. \quad (24)$$

Since $\sigma' + 2p' = \sigma'(1 + 2p'/\sigma')$, defining the parameter $\beta^2(\sigma) \equiv dp/d\sigma = p'/\sigma'$, we have that $\sigma' + 2p' = \sigma'(1 + 2\beta^2)$. Using again that $\sigma' = -2(\sigma + p)/a$, we obtain

$$V''(a) = -\frac{4M}{a^3} + \frac{6Q^2}{a^4} - 8\pi^2 [(\sigma + 2p)^2 + 2\sigma(\sigma + p)(1 + 2\beta^2)]. \quad (25)$$

Replacing in $a = a_0$, we have that $V(a_0) = V'(a_0) = 0$, so

$$V(a) = \frac{1}{2}V''(a_0)(a - a_0)^2 + O[(a - a_0)^3], \quad (26)$$

with

$$V''(a_0) = \frac{-2}{a_0^4} \left[\frac{a_0 [(a_0 - M)^3 + M(M^2 - Q^2)]}{a_0^2 - 2Ma_0 + Q^2} + 2(a_0^2 - 3Ma_0 + 2Q^2)\beta_0^2 \right], \quad (27)$$

where $\beta_0 = \beta(\sigma_0)$.

The wormhole is stable if and only if $V''(a_0) > 0$. The curves where $V''(a_0) = 0$ correspond to

$$\beta_0^2 = f_0(a_0) \equiv \frac{-a_0 [(a_0 - M)^3 + M(M^2 - Q^2)]}{2(a_0^2 - 2Ma_0 + Q^2)(a_0^2 - 3Ma_0 + 2Q^2)}, \quad (28)$$

and $a_0 = 3M/2$ ($\beta_0^2 \in \mathbb{R}$) when $|Q| = 3M/\sqrt{8}$.

For the analysis of the stability regions it is convenient to consider five cases accordingly to the value of charge:

1. Case $0 \leq \frac{|Q|}{M} < 1$. There are two regions of stability,

i) $\beta_0^2 > f_0(a_0)$ if $1 + \sqrt{1 - \frac{Q^2}{M^2}} < \frac{a_0}{M} < \frac{3}{2} + \frac{1}{2}\sqrt{9 - 8\frac{Q^2}{M^2}}$, and

ii) $\beta_0^2 < f_0(a_0)$ if $\frac{a_0}{M} > \frac{3}{2} + \frac{1}{2}\sqrt{9 - 8\frac{Q^2}{M^2}}$.

2. Case $\frac{|Q|}{M} = 1$. There are two regions of stability,

i) $\beta_0^2 > f_0(a_0)$ if $1 < \frac{a_0}{M} < 2$, and

ii) $\beta_0^2 < f_0(a_0)$ if $\frac{a_0}{M} > 2$.

3. Case $1 < \frac{|Q|}{M} < \frac{3}{\sqrt{8}}$. There are two regions of stability,

$$i) \begin{cases} \beta_0^2 < f_0(a_0) & \text{if } 0 < \frac{a_0}{M} < \frac{3}{2} - \frac{1}{2}\sqrt{9 - 8\frac{Q^2}{M^2}} \\ \beta_0^2 \in \mathbb{R} & \text{if } \frac{a_0}{M} = \frac{3}{2} - \frac{1}{2}\sqrt{9 - 8\frac{Q^2}{M^2}} \\ \beta_0^2 > f_0(a_0) & \text{if } \frac{3}{2} - \frac{1}{2}\sqrt{9 - 8\frac{Q^2}{M^2}} < \frac{a_0}{M} < \frac{3}{2} + \frac{1}{2}\sqrt{9 - 8\frac{Q^2}{M^2}} \end{cases}$$

and

$$\text{ii) } \beta_0^2 < f_0(a_0) \text{ if } \frac{a_0}{M} > \frac{3}{2} + \frac{1}{2}\sqrt{9 - 8\frac{Q^2}{M^2}}.$$

4. Case $\frac{|Q|}{M} = \frac{3}{\sqrt{8}}$. There are two regions of stability,

$$\text{i) } \beta_0^2 < f_0(a_0) \text{ if } 0 < \frac{a_0}{M} < \frac{3}{2}, \text{ and}$$

$$\text{ii) } \beta_0^2 < f_0(a_0) \text{ if } \frac{a_0}{M} > \frac{3}{2}.$$

5. Case $\frac{|Q|}{M} > \frac{3}{\sqrt{8}}$. There is one region of stability,

$$\beta_0^2 < f_0(a_0) \text{ if } \frac{a_0}{M} > 0.$$

The regions of stability for different values of charge are shown in Figs. 1 and 2. For ordinary matter, β_0 represents the velocity of sound, so it should satisfy $0 < \beta_0^2 \leq 1$. If the matter is exotic, as it happens to be in the throat, β_0 is not necessarily the velocity of sound and it is not clear which values it can take (see discussion in [6]).

For $0 < |Q|/M \leq 1$ the stability regions are similar to the case of uncharged wormholes. The *tongue* shaped region with $\beta_0^2 > 0$ moves slowly to lower values of β_0^2 and a_0/M as the charge increases. Values of β_0^2 in the interval $(0, 1]$ are not included in the stability region unless $|Q|/M$ is very close to one. The other region of stability only moves slowly to lower values of a_0/M for higher values of charge.

The situation changes dramatically if $1 < |Q|/M \leq 3/\sqrt{8}$. In this case, for every value of the parameter β_0^2 there are values of the radius a_0/M that makes the wormhole stable. When $1 < \frac{|Q|}{M} < \frac{3}{\sqrt{8}}$ and $\frac{a_0}{M} = \frac{3}{2} - \frac{1}{2}\sqrt{9 - 8\frac{Q^2}{M^2}}$, it is stable regardless of the equation of state of the exotic matter (which determines β_0^2). If $|Q|/M > 3/\sqrt{8}$, besides the large negative β_0^2 region of stability, it includes also a small one with positive β_0^2 , for radii of the throat in the range $0 < \frac{a_0}{M} < 1 + \sqrt[3]{\frac{Q^2}{M^2} - 1}$.

4 Conclusions

We have applied a linearization stability analysis to thin-shell wormholes endowed with charge and we have found that the presence of large charges ($|Q|/M \approx 1$), significantly increases the possibility of obtaining stable wormhole spacetimes, imposing less severe restrictions on the equation of state of the exotic matter that must be present in the wormhole throat. Although it is not possible to give a full physical meaning to the parameter β_0 in the absence of a detailed microphysical model for the exotic matter, our analysis shows that, for a given radius a_0 , it is *always* possible to find stable traversable wormhole solutions for *any* value of β_0 by taking adequate values of charge and mass.

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