The equilibrium limit of the Casati-Prosen model

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An alternative explanation of the decoherence in the Casati-Prosen model is presented. It is based on the Self Induced Decoherence formalism extended to non-integrable systems.

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I. INTRODUCTION.

The Casati-Prosen model [1] combines two paradigmatic models of classical and quantum mechanics: a Sinai billiard, where the simplest examples of chaotic motion take place and a Young, two slits, experiment, the main example of quantum behavior that it "...is impossible absolutely impossible to explain in any classical way" [2]. So we really could call this model the "Sinai-Young" experiment. We consider that the complete understanding of this model is essential to solve problems like quantum irreversibility, decoherence, and chaos. The model is shown in figure 1 (of paper [1]), namely a triangular *upper billiard* with perfectly reflecting layers, with two slices in its base, on the top of a box, the *radiating region*, with a photographic film in its base and absorbent walls. A quantum state with a gaussian packet initial condition bounces in the triangle, and produces two centers of radiation in the two slices from which a small amount of probability current leaks from the billiard to the radiating zone. Then when the billiard is perfectly triangular and therefore integrable (full-lines in figure 1 of [1]) the interference fringes (full-lines in figure 2) appear in the film and when it is a Sinai billiard and therefore non-integrable (dotted line of fig 1) the first pattern decoheres to the (dotted) curve of figure 2. This computer experiment shows how complexity can produce decoherence (without an environment or an external noise) and it is explained in paper [1] using a kinematical average. As the subject is so important we would like to add another feature to the Casati and Prosen explanation of the phenomenon showing that the model reaches an equilibrium state where decoherence appears. In doing so we will use our previous results on decoherence [3], mainly paper [4], where local constants of the motion are introduced both at the classical and quantum level allowing to define non-integrable quantum systems and to give a minimal definition of quantum chaos, and paper [5], where decoherence times are found. Recently we have shown [6] that our formalism, "Self Induced Decoherence" (SID) can be encompassed with the traditional one "Environment Induced Decoherence" (EID) [7], combining the advantages of both formalisms.

II. THE PROBLEMS OF PAPER [1].

To make clear our physical point of view let us consider the two main problems to understand chaotic motion in terms of quantum mechanics listed in the introduction of paper [1] (see also [8]):

1.- How is it possible to find chaos in bound systems, with finite number of particles, which have a quasi-periodic behavior and therefore a discrete evolution spectrum, if chaotic (e. g. mixing) motion requires a continuous one?

We consider that the solution can be found in paper [9] where it is shown that, even if a quantum system has a discrete evolution spectrum, the motion can be modeled with a continuous spectrum for times much smaller than recurrence or Poincaré time. For a discrete energy spectrum $\{\alpha_\nu\}$ this time is

$$t_P \approx \frac{2\pi\hbar}{\min(\alpha_{\nu+1} - \alpha_\nu)} \tag{1}$$

so if the distances among the eigenvalues are very small t_P is extremely large. Then for $t \ll t_P$ the typical theorems, e.g. the Riemann-Lebesgue theorem, can be used.

2.- In quantum motions initial errors propagate linearly while in chaotic system this propagation is exponential. This contradiction makes quantum chaos impossible.

We consider that, most likely, this kind of reasonings is done in quantum systems with an integrable classical system as classical limit. If this is not the case (as in the systems studied in [4]) it can be demonstrated that the trajectories in the classical limit are chaotic and may have positive Lyapunov exponents. So the contradiction is solved.

III. THE BASES OF OUR ALTERNATIVE EXPLANATION.

We will give our alternative explanation based in three results. In this section we only give a sketch of the main ideas on these subjects, the complete treatment and figures can be found in the references:

a.- In paper [4], using the Weyl-Wigner-Moyal isomorphs the definition of classical integrable and non-integrable system is extended to the quantum case. Then the SID formalism is extended to the non-integrable system. For N configuration variables these systems, in the classical case have less than N global constants of the motion. But according to the Carathéorory-Jacobi lemma [10] they have N constants of the motion locally defined which, via a Weyl-Wigner-Moyal isomorphism, allow to define N local Complete System of Commuting Observables that are used in the extension of SID. The resulting theory is very similar to the original one. Only an extra index i corresponding to the domain D_{ϕ_i} (that contains the point of the phase space ϕ_i and where local constant of motion are defined) must be added in all summations. Then the state of the system $\rho(t)$ reaches an equilibrium state ρ_*, given by eq. (3.23) of [4], defined as a weak limit

$$W\lim_{t\to\infty}\rho(t)=\rho_*=\sum_{imm'}\int_0^\infty d\omega\rho(\omega)_{\phi_i p}(\omega,m,m'|_{\phi_i} \tag{2}$$

where ω is the eigenvalue of the hamiltonian H (which is considered to be globally defined), $m_{\phi_i}=(m_x,m_y)_{\phi_i}$ in our case will be the eigenvalue of the local momentum $\mathbf{P}_{\phi_\mathbf{i}}=(P_x,P_y)_{\phi_i}$, and $(\omega,m,m'|_{\phi_i}$ the cobasis of the eigen basis of the CSCO $\{H,\mathbf{P}_{\phi_\mathbf{i}}\}$, namely $\{|\omega,m\rangle_{\phi_i}\langle\omega,m'|_{\phi_i}\}$. Then the equilibrium final state ρ_* has decohered in the energy since only the diagonal terms in ω appear (if not the basis would be $(\omega,\omega',m,m'|_{\phi_i})$ but not in the remaining observables $(P_x,P_y)_{\phi_i}$, since non-diagonal terms m, m' do appear. Then via a simple diagonalization in the indices m,m' we reach to eq. (3.33)

$$W\lim_{t\to\infty}\rho(t)=\rho_*=\sum_{ip}\int_0^\infty d\omega\rho(\omega)_{\phi_i p}(\omega,p,p|_{\phi_i} \tag{3}$$

where $p_{\phi_i}=(p_x,p_y)_{\phi_i}$ are the eigenvalues of an adequate CSCO $\{H,\mathbf{O}_{\phi_\mathbf{i}}\}$. In the correspondent eigenbasis ρ_* is fully decohered, since now only diagonal terms (in ω and p) appear.

b.- The upper triangle will be considered as the Sinai billiard of appendix A of paper [4]. Namely the triangle will be complemented by three potential walls in such a way that these potentials $U(x,y)$ (similarly to those of the Sinai billiard of appendix A of paper [4]) produce the bounces against the sides of the triangle. We will call D_0 the interior of the triangle (as the D_0 of figure 2 of [4]). It has two independent local constants of the motion: H and P_x (or H and P_y or $P_x$and P_y since $H=\frac{1}{2M}(\ P_x^2+\ P_y^2)$). Then we will add three extra domains, each one for each potential wall, D_1,D_2,D_4. In the case of the triangle with straight sides the local constants of the motion in the boundaries can be deduced by their symmetries. They are:

- In the horizontal boundary ($U(x,y)=U_1(y)$, domain D_1) H and P_x.
- In the vertical boundary ($U(x,y)=U_2(x)$, domain D_2) H and P_y
- In the third boundary ($U(x,y)=U_4(ax+yb)$ domain D_4) H and a linear combination of P_x and P_y.

This is not the case if the triangle has a circular boundary (with radius $r=a$ and angular coordinate θ,where the constants of the motion in the third boundary ($U_4(x,y)=U(r)$, domain D_4) are H and P_θ.

c.- Decoherence times t_D will be calculated using references [5] and [11]. From reference [5] we know that

$$t_D=\frac{\hbar}{\gamma}$$

where γ is the distance to the real axis of the pole of the resolvent closer to this axis. These poles for a circular symmetric potential can be computed from reference [11] From eqs. (5.1.4) and (5.5.24) of this reference we know that the energy is

$$E=\frac{\hbar^2k^2}{2M},\ \text{and}\ k=\frac{\beta}{a}$$

where M is the mass, being, from eq. (5.5.29), the β for the pole closer to the real axis

$$\beta_0 = R_0 - iI_0 = U_0 - \left(\frac{m+2}{4U_0}\right)\ln\left(\frac{2U_0^{m+2}}{A^2}\right) - \frac{i}{2}\ln\left(\frac{2U_0^{m+2}}{A^2}\right)$$

where $U^{(m)}(a-)$ is the first non vanishing derivative of the potential at the boundary (corresponding to the side of the potential) and coefficients U_0 and A are given by eqs. (5.5.24) and (5.5.26) of [11].

Then

$$\gamma = \frac{\hbar^2 R_0 I_0}{2Ma^2}, \text{ and } t_D = \frac{2Ma^2}{\hbar R_0 I_0} \tag{4}$$

Below we will use this equation.

IV. THE TRIANGLE WITH STRAIGHT SIDES.

Let us first consider the case of the straight triangle and let us take as initial condition in the triangle a pure state wave packet $|\varphi\rangle \sim |\varphi(\mathbf{x},0)\rangle$ (which of course it is not an eigenstate of the momentum operator $\mathbf{P}$). With this initial condition we obtain the solution $|\varphi\rangle \sim |\varphi(\mathbf{x},t)\rangle$ in the triangle, that can be written as a matrix

$$\rho(t) = |\varphi(\mathbf{x},t)\rangle\langle\varphi(\mathbf{x},t)| \tag{5}$$

We can make some remarks:

i.- If the billiard is considered classical, two initial parallel trajectories remain parallel while they bounce in the triangle. Therefore there are neither positive Lyapunov exponent nor chaos.

ii.- Even if according to eqs. (2) or (3) there will be decoherence in an infinite time, in this case the characteristic decoherence time is, in fact, infinite since all the potential walls in this case are straight lines and therefore the radius $a \to \infty$, then from eq. (4) $t_D \to \infty$. Therefore $\rho(t)$ remains bouncing forever in the triangle and does not decohere.

Let us now consider the lower part under the slit screen.. The direct impact of the packet (5) produces two boundary conditions in the two slits:. These two boundary conditions produce two circular-symmetric solutions, $|\varphi_1(\mathbf{x},t)\rangle$ and $|\varphi_2(\mathbf{x},t)\rangle$, with centers of symmetry in the two slits. Therefore the state in the lower part is $|\varphi(\mathbf{x},t)\rangle = |\varphi_1(\mathbf{x},t)\rangle + |\varphi_2(\mathbf{x},t)\rangle$ and the probability at $\mathbf{x}$ is:

$$p = \langle|\mathbf{x}\rangle\langle\mathbf{x}|\rangle_{|\varphi(\mathbf{x},t)\rangle\langle\varphi(\mathbf{x},t)|} = p_1 + p_2 + p_{int}$$

where

$$p_1 = |\varphi_1(\mathbf{x},t)|^2 \geq 0, \; p_2 = |\varphi_2(\mathbf{x},t)|^2 \geq 0, \quad p_{int} = 2\,\mathrm{Re}(\varphi_1(\mathbf{x},t)\varphi_2^*(\mathbf{x},t)) \tag{6}$$

Of course $p_{int} \neq 0$ and it is the interference term. Let us observe that as $|\varphi(\mathbf{x},t)|^2$ is time invariant p is also time invariant as it is verified in [1] figure 4a.

Moreover if we consider many bounces of the packet instead of just the direct impact, instead of (5) we will have a sum with different momenta $\mathbf{P}$. But if the system is integrable this sum will have a finite number of terms (see [1] and [12]) and the interference fringes will remain.

V. THE TRIANGLE WITH A CURVED SIDE.

Let us now consider the case of the curved triangle. Now

i.- Initial parallel trajectories will lose their parallelism when they collide with the curved side and there will be positive Lyapunov exponent and chaos. In fact, Sinai billiards are K-systems.

ii.- The potential walls are not trivial (i.e. $a \neq \infty$) and therefore the analytic continuation of the resolvent has complex poles and the finite decoherence time is given by eq. (4)[1].

[1]Essentially, as explained, the system has a central domain D_0 and three potential boundaries domains D_1, D_2, and D_4. But each scattering in the potential of D_4 can be considered as beginning in the domain $D_0^- = D_{\phi_1}$ and ending in the out domain $D_0^+ = D_{\phi_2}$. In each scattering the values of the constants of the motion change. As this scattering is repeated again and again really D_0 must be considered as an infinite sequence of D_{ϕ_i}

Then in this case, taking into account the caveat of section II.1, we can consider that since the system is a K-system, it has a continuous spectrum, so using the results reviewed in sections III we can say that the state $\rho(t)$ inside the billiard reaches an equilibrium limit ρ_* given by eq. (2). Now, $(\omega, p, p'|_{\phi_i}$ is a functional (i.e. a distribution or kernel) in the continuous variable ω but it is a trivial matrix in the discrete variables p, p'. Nevertheless, based on the observation of section II.1, we can consider ω as a discrete variable that has been approximated by a continuous one so we can substitute $(\omega, p, p'|_{\phi_i}$ by $|\omega, p\rangle_{\phi_i}\langle\omega, p'|_{\phi_i}$ then the eq. (2) reads

$$W\lim_{t\to\infty}\rho(t) = \rho_* = \sum_{ip\omega}\rho(\omega)_{\phi_i p}|\omega, p\rangle_{\phi_i}\langle\omega, p|_{\phi_i} \tag{7}$$

This is the equilibrium state of the upper part that substitutes the $\rho(t) = |\varphi(\mathbf{x},t)\rangle\langle\varphi(\mathbf{x},t)|$ of eq. (5). Now we must obtain the corresponding solution in the lower part, solving the von Neumann equation. But this equation is linear, and now the initial conditions are provided not by (5) but by (7), and since in (7) ρ_* is a linear combination of $|\omega, p\rangle_{\phi_i}\langle\omega, p'|_{\phi_i}$, to obtain the new p_{int} we must only repeat the same linear combination, e.g.

$$p_{int} = 2\sum_{ip\omega}\rho(\omega)_{\phi_i p}\,\mathrm{Re}(\varphi_{1\omega p}(\mathbf{x})\varphi^*_{2\omega p}(\mathbf{x}))_{\phi_i}$$

where

$$\varphi_{1\omega p}(\mathbf{x}) = \langle\mathbf{x}|\omega, p\rangle_{1\phi_i}, \qquad \varphi_{2\omega p}(\mathbf{x}) = \langle\mathbf{x}|\omega, p\rangle_{2\phi_i}$$

are the solutions in the lower box, centered in the slits 1 and 2 respectively. Now we make the inverse transformation the one that allows to go from eq. (2) to eq. (3), i.e. $|\omega, p\rangle_{\phi_i} = U^m_{p\phi_i}|\omega, m\rangle_{\phi_i}$ where $U^m_{p\phi_i}$ is the unitary transformation that diagonalizes $(\omega, m, m'|_{\phi_i}$ so

$$p_{int} = 2\sum_{ip\omega mm'}\rho(\omega)_{\phi_i p}\,\mathrm{Re}[U^m_{p\phi_i}\varphi_{1\omega m}(\mathbf{x},)(U^{m'}_{p\phi_i}\varphi_{2\omega m'}(\mathbf{x})^*]_{\phi_i} =$$

$$\sum_{ip\omega mm'}\rho(\omega)_{\omega p\phi_i}[(U^m_{p\phi_i}\varphi_{1\omega m}(\mathbf{x})(U^{m'}_{p\phi_i}\varphi_{2\omega m'}(\mathbf{x}))^* + (U^m_{p\phi_i}\varphi_{1\omega m}(\mathbf{x}))^* U^{m'}_{p\phi_i}\varphi_{2\omega m'}(\mathbf{x})]_{\phi_i}$$

Now $\varphi_{1\omega m}(\mathbf{x},t)$ and $\varphi^*_{2\omega m'}(\mathbf{x},t)$ are eigenvalues of H and P_x, and therefore also of P_y, then[2]

$$\varphi_{1\omega m}(\mathbf{x}',t) \sim e^{-i\frac{\mathbf{m}.\mathbf{x}'}{\hbar}}, \quad \varphi^*_{2\omega m'}(\mathbf{x}'',t) \sim e^{-i\frac{\mathbf{m}'.\mathbf{x}''}{\hbar}}$$

But in the slits one of this functions is obtained from the other by a displacement $\mathbf{s} = (s, 0)$ where s is the distance between the slits. Then calling $\mathbf{x}' = \mathbf{x} - \frac{1}{2}\mathbf{s}$ and $\mathbf{x}'' = \mathbf{x} + \frac{1}{2}\mathbf{s}$ we have

$$p_{int} = 2\sum_{ipp'\omega mm'}\rho(\omega)_{\phi_i pp'}\,\mathrm{Re}[U^m_{p\phi_i}e^{-i\frac{\mathbf{m}.(\mathbf{x}-\frac{1}{2}\mathbf{s})}{\hbar}}(U^{m'}_{p\phi_i})^* e^{i\frac{\mathbf{m}'.(\mathbf{x}+\frac{1}{2}\mathbf{s})}{\hbar}}]_{\phi_i} =$$

$$2\sum_{ipp'\omega mm'}\rho(\omega)_{\phi_i pp'}\,\mathrm{Re}[U^m_{p\phi_i}(U^{m'}_{p\phi_i})^* e^{-i\frac{(\mathbf{m}-\mathbf{m}').\mathbf{x}}{\hbar}}e^{i\frac{(\mathbf{m}+\mathbf{m}').\mathbf{s}}{2\hbar}}]_{\phi_i} =$$

$$\sum_{ipp'\omega mm'}\rho(\omega)_{\phi_i pp'}[U^m_{p\phi_i}(U^{m'}_{p\phi_i})^* e^{-i\frac{(\mathbf{m}-\mathbf{m}').\mathbf{x}}{\hbar}}e^{i\frac{(\mathbf{m}+\mathbf{m}').\mathbf{s}}{2\hbar}} + U^{m'}_{p\phi_i}(U^m_{p\phi_i})^* e^{i\frac{(\mathbf{m}-\mathbf{m}').\mathbf{x}}{\hbar}}e^{-i\frac{(\mathbf{m}+\mathbf{m}').\mathbf{s}}{2\hbar}})_{\phi_i}$$

Now we can rephrase what we have said in section II.1 but now related to the discrete variable $\mathbf{m}$ instead of α_ν. Now the number of terms in the summation is extremely big. In fact, while in the integrable case there will be just a

[2]Really these solutions must be added in order to satisfy the boundary condition of the lower part of the system, but this is just another summation that does not modify the final result.

few terms (see end of section IV), now the system is not integrable and the terms may be infinite, since they arrive from every direction[3], so these $\mathbf{m}$ are very close. Thus the last $\sum_{ipp'mm'}$ can be considered as two integrals in the $\mathbf{m}$ and in the $\mathbf{m}'$ that can be changed in two integrals in the $\mathbf{m}+\mathbf{m}'$ and the $\mathbf{m}-\mathbf{m}'$. In particular the integrals contain the factors $e^{i\frac{(\mathbf{m}-\mathbf{m}').\mathbf{x}}{2\hbar}}, e^{-i\frac{(\mathbf{m}-\mathbf{m}').\mathbf{x}}{2\hbar}}$. So as there is a macroscopic distance from the two slits screen to the photographic plate $\mathbf{x}$ is macroscopic with respect to $\hbar$ in such a way that we can consider that $\frac{\mathbf{x}}{\hbar}\to\infty$ and we can use the Riemann-Lebesgue theorem concluding that

$$p_{int}=\sum_{ipp'^{\omega}mm'}\rho(\omega)_{\phi_i pp'}[U_{p\phi_i}^{m}(U_{p\phi_i}^{m'})^{*}e^{-i\frac{(\mathbf{m}-\mathbf{m}').\mathbf{x}}{\hbar}}e^{i\frac{(\mathbf{m}+\mathbf{m}').\mathbf{s}}{2\hbar}}+U_{p\phi_i}^{m'}(U_{p\phi_i}^{m})^{*}e^{i\frac{(\mathbf{m}-\mathbf{m}').\mathbf{x}}{\hbar}}e^{-i\frac{(\mathbf{m}+\mathbf{m}').\mathbf{s}}{2\hbar}})_{\phi_i}=0$$

So the interference fringes vanish and there is decoherence in the final equilibrium state. q. e. d.

VI. CONCLUSION

1.- We have shown that the Casati-Prosen model reaches an equilibrium state in a finite decoherence time. In this final state the interference fringes vanish and we have decoherence. From eq. (4), taking for M the electron mass and $a=1cm$ we have $t_D\approx 1s$.

2.- There is no environment, decoherence is produced by complexity. So the computational result of Casati and Prosen cannot be explained by EID. But, we have demonstrated in [6] that a new combined formalism can encompass, in a consistent way, EID and SID. In this case SID solves a problem that cannot be solved by EID. The conclusions are that EID is a correct theory but it is incomplete and that it can be completed with SID.

3.- All the reasoning has being done at the quantum level (with some side remarks at the classical level) so we may say that the decoherence is produced by quantum chaos. We will try to precise this notion based in SID formalism in the near future.

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