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Relativistic dynamics, Green function and pseudodifferential operators

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The central role played by pseudodifferential operators in relativistic dynamics is known very well. In this work, operators like the Schrodinger one (e.g., square root) are treated from the point of view of the non-local pseudodifferential Green functions. Starting from the explicit construction of the Green (semigroup) theoretical kernel, a theorem linking the integrability conditions and their dependence on the spacetime dimensions is given. Relativistic wave equations with arbitrary spin and the causality problem are discussed with the algebraic interpretation of the radical operator and their relation with coherent and squeezed states. Also we perform by means of pure theoretical procedures (based in physical concepts and symmetry) the relativistic position operator which satisfies the conditions of integrability: it is a non-local, Lorentz invariant and does not have the same problems as the "local"position operator proposed by Newton and Wigner. Physical examples, as zitterbewegung and rogue waves, are presented and deeply analyzed in this theoretical framework. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4953368]

I. INTRODUCTION

Since long time ago, the correct mathematical treatment of various operators that contain a nonlinear character called the permanent attention of researchers in both physics and mathematics. In particular, the natural introduction of the square root of the fundamental operators (Hamiltonian, Lagrangian) in the field of relativistic theories has brought not only the problem of mathematical treatment of these non-local and nonlinear operators, but its physical interpretation also. The conceptual fact to find the physical interpretation of the square root operator has not only addressed the classical dynamics but in particular its action on physical states of the system, mainly in the quantum case (spectrum). About this issue, the works of Salpeter and others[22] are very well known. From the purely formal point of view, the mathematical developments have been oriented in the area of the pseudodifferential operators with the works of Ref. 24 and in the theory of semigroups, mainly with the research of Yosida.[25]

Several different physical scenarios have been shown in which the correct description is made by means of these pseudodifferential operators. For example, in Ref. 23 a semi-analytical computation of the three dimensional Green function of a pseudodifferential operator for seakeeping flow problems is proposed where the potential flow model is assumed with harmonic dependence on time and a linearized free surface boundary condition. Also in Ref. 26, the pseudodifferential operator is introduced in Carson's integral computation (the original expression involves the evaluation of just one Struve and one Bessel function) and used in power systems analysis for evaluating the earth-return impedance of overhead conductors above homogeneous earth.

The starting point in this article is to analyze the classical radical operator from the viewpoint of Green functions connecting these results with previous investigations in which we have related these types of operators with the well known problem of localization and proper time. In the second place, we will show the purely algebraic representation giving an interpretation of such operators.

This algebraic interpretation is very important because it brings the possibility to make a link with pseudodifferential operators and semigroup (Fourier-Integral) representations:

$$\begin{array}{ccc} & algebraic & \\ & interpretation & \\ \nearrow & & \nwarrow \\ pseudodifferential\ operators & \longleftrightarrow & \begin{array}{c} semigroup \\ (Fourier-Integral\ representations) \end{array} \end{array}$$

and consequently with the relativistic wave equations of arbitrary spin states (in particular of parastatistical ones). Several examples, from our earlier references on the subject (e.g., Refs. 29 and 44), and new ones (as the important problem of rogue waves) are analyzed and discussed. Finally, we discuss other important questions that will be treated in detail somewhere[27] as the role of the spin, the dispersion in time of the physical states, and the Levy processes.[49]

II. GREEN FUNCTIONS AND PSEUDODIFFERENTIAL OPERATORS

The Schrodinger operator is, in general, the main object of study both: in theoretical physics and mathematics and from the classical to the quantum point of view. However, from the operator viewpoint, the relativistic form (e.g., square root) carry conceptual and technical troubles. As we have discussed before,[1,2,34] the conceptual trouble coming from three sources: the meaning of the Lagrangian as *measure*, the *localization* (one particle/ensemble interpretation), and the relation with *spin degrees of freedom*; meanwhile, the technical one is given by the square root form of the Lagrangian/Hamiltonian when is treated as *operator*.[3] Consequently, in this paper our starting point will be the following simple mathematical object (principal kernel):

$$\mathbb{L} \equiv \sqrt{\zeta^2 + m^2}, \tag{1}$$

that is, characteristic inside the fundamental Schrodinger type equation to solve ($g_{\mu\nu} = (+---)$; μ, $\nu = 0,1,2,3$)), namely,

$$i\partial_t \psi(t,x) = \frac{1}{(2\pi)^3} \int\int \sqrt{\zeta^2 + m^2} e^{i\zeta\cdot(x-y)} \psi(t,y)\, d^3y\, d^3\zeta \equiv \mathcal{H}\psi(t,x) \tag{2}$$

and in having account that the square root operator above is well defined in such pseudodifferential representation due to its strong ellipticness (for instance, $\zeta^2 \equiv \gamma_{ij}\zeta^i\zeta^j$ with $i,j = 1,2,3$), the solution for the source which is a delta function becomes

$$i\partial_t G(t,x) - \frac{1}{(2\pi)^3} \int\int \sqrt{\zeta^2 + m^2} e^{i\zeta\cdot(x-y)} G(t,y)\, d^3y\, d^3\zeta \equiv -\delta^3(x)\,\delta(t). \tag{3}$$

We perform the Fourier transform obtaining

$$\frac{1}{(2\pi)^2} \int (\omega G(\omega,p) - \sqrt{p^2+m^2} G(\omega,p)) e^{-i(\omega t - p\cdot x)} d^3p\, d\omega = -\frac{1}{(2\pi)^4} \int e^{-i(\omega t - p\cdot x)} d^3p\, d\omega, \tag{4}$$

consequently we have in the momentum space

$$G(\omega,p) = -\frac{1}{(2\pi)^2} \frac{1}{\omega - \sqrt{p^2+m^2}}. \tag{5}$$

Anti-transforming back to $x-$space we obtain

$$G(t,x) = -\frac{1}{(2\pi)^4} \int \frac{e^{-i(\omega t - p\cdot x)}}{\omega - \sqrt{p^2+m^2}} d^3p\, d\omega. \tag{6}$$

Choosing a path over the complex plane, the integration over ω can be performed. Then we replace the denominator by $\omega - (1 \mp i\varepsilon)\sqrt{p^2+m^2}$ with $\varepsilon > 0$ evidently. Integration along a in the upper half plane gives $G^+(t,x)$ and the other one $G^-(t,x)$

$$G^{+}(t,x)=\frac{i}{(2\pi)^{3}}\lim_{\varepsilon\to 0}\theta(t)\int e^{-i\left((1-i\varepsilon)\sqrt{p^{2}+m^{2}}t-p\cdot x\right)}d^{3}p, \tag{7}$$

$$G^{-}(t,x)=-\frac{i}{(2\pi)^{3}}\lim_{\varepsilon\to 0}\theta(-t)\int e^{-i\left((1+i\varepsilon)\sqrt{p^{2}+m^{2}}t-p\cdot x\right)}d^{3}p, \tag{8}$$

where $\theta(t)$ is the Theta (step) function. Notice that $G_D=G^{+}(t,x)-G^{-}(t,x)=\frac{i}{(2\pi)^{3}}\int e^{-i\left(\sqrt{p^{2}+m^{2}}t-p\cdot x\right)}d^{3}p$.

Integration with respect to d^3p is performed as follows: we write in the spherical prescription only for simplicity, $p\cdot x=|p|\,|x|\cos\theta$ and integrating with respect to φ and θ

$$G^{\pm}(t,x)=\pm\frac{i}{(2\pi)^{3}}\lim_{\varepsilon\to 0}\theta(\pm t)\int e^{-i\left((1\mp i\varepsilon)\sqrt{p^{2}+m^{2}}t-|p||x|\cos\theta\right)}p^{2}dp\sin\theta d\theta d\varphi \tag{9}$$

$$=\pm\frac{2i}{(2\pi)^{2}}\lim_{\varepsilon\to 0}\theta(\pm t)\int e^{-it(1\mp i\varepsilon)\sqrt{p^{2}+m^{2}}}pdp\sin(|p|\,|x|) \tag{10}$$

$$=\pm\frac{2i}{(2\pi)^{2}}\lim_{\varepsilon\to 0}\theta(\pm t)\,it(1\mp i\varepsilon)\,m^{2}\frac{K_{2}\left(m\sqrt{x^{2}-t^{2}(1\mp i\varepsilon)^{2}}\right)}{x^{2}-t^{2}(1\mp i\varepsilon)^{2}} \tag{11}$$

where we have used Ref. 28 (formulas 3.915).

III. RELATION WITH THE OPERATIONAL APPROACH: EXAMPLE IN ONE DIMENSION

From the fundamental equation to solve ($g_{\mu\nu}=(+---)\,;\mu,\nu=0,1,2,3)$),

$$i\partial_{t}\psi(t,x)=\frac{1}{(2\pi)^{3}}\int\int\sqrt{\zeta^{2}+m^{2}}e^{i\zeta\cdot(x-y)}\psi(t,y)\,dy\,d\zeta\equiv\mathcal{H}\psi(t,x), \tag{12}$$

and in having account that the square root operator above is well defined in such pseudodifferential representation due to its strong ellipticness (for instance, $\zeta^{2}\equiv\gamma_{ij}\zeta^{i}\zeta^{j}$ with $i,j=1,2,3$), the solution for the source becomes

$$i\partial_{t}G(t,x)-\frac{1}{(2\pi)^{3}}\int\int\sqrt{\zeta^{2}+m^{2}}e^{i\zeta\cdot(x-y)}G(t,y)\,dy\,d\zeta\equiv-\delta(x)\,\delta(t). \tag{13}$$

We perform the Fourier transform obtaining

$$\int(\omega G(\omega,p)-\sqrt{p^{2}+m^{2}}G(\omega,p))e^{-i(\omega t-p\cdot x)}dpd\omega=-\frac{1}{(2\pi)^{2}}\int e^{-i(\omega t-p\cdot x)}dpd\omega, \tag{14}$$

consequently we have in the momentum space

$$G(\omega,p)=-\frac{1}{\omega-\sqrt{p^{2}+m^{2}}}. \tag{15}$$

Anti-transforming back to $x-$space we have

$$G(t,x)=-\frac{1}{(2\pi)^{2}}\int\frac{e^{-i(\omega t-p\cdot x)}}{\omega-\sqrt{p^{2}+m^{2}}}dpd\omega. \tag{16}$$

Again, choosing a path over the complex plane, the integration over ω can be performed. Then, we replace the denominator by $\omega-(1\mp i\varepsilon)\sqrt{p^{2}+m^{2}}$ with $\varepsilon>0$ as is evident. Integration along a in the upper half plane gives $G^{+}(t,x)$ and the other one $G^{-}(t,x)$

$$G^{+}(t,x)=\frac{i}{2\pi}\lim_{\varepsilon\to 0}\theta(t)\int e^{-i\left((1-i\varepsilon)\sqrt{p^{2}+m^{2}}t-p\cdot x\right)}dp, \tag{17}$$

$$G^{-}(t,x)=-\frac{i}{2\pi}\lim_{\varepsilon\to 0}\theta(-t)\int e^{-i\left((1+i\varepsilon)\sqrt{p^{2}+m^{2}}t-p\cdot x\right)}dp, \tag{18}$$

where $\theta(t)$ is the Theta (step) function. Notice that $G^{+}(t,x)-G^{-}(t,x)=\frac{i}{2\pi}\int e^{-i\left(\sqrt{p^{2}+m^{2}}t-p\cdot x\right)}dp$.

Integration with respect to dp is performed directly obtaining

$$G^{\pm}(t,x) = \pm\frac{i}{2\pi}\lim_{\varepsilon\to 0}\theta(\pm t)\, it\,(1\mp i\varepsilon)\, m\frac{K_1\left(m\sqrt{x^2-t^2(1\mp i\varepsilon)^2}\right)}{\sqrt{x^2-t^2(1\mp i\varepsilon)^2}} \tag{19}$$

where we have used Ref. 28 (formulas 3.914).

Now we will make the proof about the direct relation between the operational approach and the Green function one.

We know that

$$LG(t,x) \equiv i\partial_t G(t,x) - \frac{1}{(2\pi)^3}\int\int\sqrt{\zeta^2+m^2}e^{i\zeta\cdot(x-y)}G(t,y)\,dy\,d\zeta \tag{20}$$

$$\equiv -\delta(x)\,\delta(t) \tag{21}$$

then, the solution given by the difference of the fundamental Green functions is the solution of the free Salpeter equation with constant initial condition

$$LG_D = L\left(G^{+}(t,x) - G^{-}(t,x)\right) = 0, \tag{22}$$

multiplication with the general initial condition as (a is some constant)

$$\Psi_0(p) \equiv a\int e^{-i\left(\sqrt{p'^2+m^2}\beta - p'\alpha\right)}\delta(p-p')\,dp' \tag{23}$$

is also solution. It is useful to note that $\Psi_0(p)$ acts as convolution due to its reproducing properties (notice the obvious fact that have the same fashion that the kernel for the square root transformation as in the Schwartzian case) making a general shift in time and space

$$\left(G^{+}(t,x) - G^{-}(t,x)\right)\Psi_0(p) = \frac{i}{2\pi}\int e^{-i\left(\sqrt{p^2+m^2}(t+\beta)-p(x+\alpha)\right)}dp = 0 \tag{24}$$

$$= \frac{1}{2\pi}(t+\beta)\, m\frac{K_1\left(m\sqrt{(x+\alpha)^2-(t+\beta)^2}\right)}{\sqrt{(x+\alpha)^2-(t+\beta)^2}}. \tag{25}$$

Then, the suitable initial condition for the vacuum acts also as potential in the Green's method (for any $G^{\pm}(t,x)$)

$$\int(\omega G^{\pm}(\omega,p) - \sqrt{p^2+m^2}G^{\pm}(\omega,p))e^{-i(\omega t - p\cdot x)}e^{-i\left(\sqrt{p^2+m^2}\beta - p\alpha\right)}dp\,d\omega = \tag{26}$$

$$= -\frac{1}{(2\pi)^2}\int e^{-i(\omega t - p\cdot x)}dp\,d\omega\, e^{-i\left(\sqrt{p^2+m^2}\beta - p\alpha\right)}.$$

IV. THE DIMENSION-DEPENDENCE OF THE GREEN FUNCTION

Theorem 1. *The Green function of the square root pseudodifferential operator in the case of even dimensional spacetime is exactly integrable being expressed in a closed form as derivatives of the Mac Donald's function K_2 (Modified Bessel Function of the Second Kind).*

Proof. Following the procedure previously introduced, we introduce hyperspherical coordinates as usual

$$x_1 = r\cos\phi_1, \tag{27}$$

$$\begin{aligned} x_2 &= r\sin\phi_1\cos\phi_2, \\ x_3 &= r\sin\phi_1\sin\phi_2\cos\phi_3, onumber \\ &\vdots \\ x_N &= r\sin\phi_1\sin\phi_2\sin\phi_3\cdots\sin\phi_{N-1}\cos\phi_N, \end{aligned} \tag{28}$$

consequently, our problem will be the following integral:

$$\int e^{-i\left((1\mp i\varepsilon)\sqrt{p^2+m^2}t-|p||x|\cos\theta_1\right)}p^{N-1}dp\sin\theta_1 d\theta_1\sin\theta_2 d\theta_2\cdots\sin\theta_{N-1}d\theta_{N-1}d\varphi = \tag{29}$$
$$= F\underbrace{\int e^{-i\left((1\mp i\varepsilon)\sqrt{p^2+m^2}t-|p||x|\cos\theta_1\right)}p^{N-1}dp\sin\theta_1 d\theta_1}_{\equiv I},$$

where is a numerical factor coming from the: $\sin\theta_2 d\theta_2\cdots\sin\theta_{N-1}d\theta_{N-1}d\varphi$ integration

$$F = \int \sin\theta_2 d\theta_2\cdots\sin\theta_{N-1}d\theta_{N-1}d\varphi = 2\pi 2^{N-2}. \tag{30}$$

Let us to compute the I integral, integrating in θ_1 first

$$\begin{aligned} I &= \int e^{-i\left((1\mp i\varepsilon)\sqrt{p^2+m^2}t-|p||x|\cos\theta_1\right)}p^{N-1}dp\,(-d\cos\theta_1) \\ &= \int e^{-i(1\mp i\varepsilon)\sqrt{p^2+m^2}t}p^{N-1}dp\left(-\frac{e^{-i|p||x|\cos\theta_1}}{i\,|p|\,|x|\cos\theta_1}\bigg|_{\theta_i=0}^{\theta_f=\pi}\right) \\ &= \int e^{-i(1\mp i\varepsilon)\sqrt{p^2+m^2}t}\frac{p^{N-1}}{i\,|p|\,|x|}dp\left(-e^{-i|p||x|}+e^{i|p||x|}\right) \\ &= \int e^{-i(1\mp i\varepsilon)\sqrt{p^2+m^2}t}\frac{p^{N-2}}{|x|}dp2\sin\left(|p|\,|x|\right). \end{aligned} \tag{31}$$

We have been solved the case for $d = 2(N = 1)$ and $d = 4(N = 3)$. We can demonstrate the integrability of the higher dimensional family of cases where the spacetime dimension is even $d = 1 + N$. The starting point is to use the formula

$$p^{N-2}\sin\left(|p|\,|x|\right) = \left(\frac{id}{dx}\right)^{N-3}\left(p\sin\left(|p|\,|x|\right)\right), \tag{32}$$

for example,

$$p^{5-2}\sin\left(|p|\,|x|\right) = -\frac{d^{5-3}}{dx}\left(p\sin\left(|p|\,|x|\right)\right)$$
$$p^3\sin\left(|p|\,|x|\right) = -\frac{d^2}{dx}\left(p\sin\left(|p|\,|x|\right)\right) = -p^2\frac{d}{dx}\left(\cos\left(|p|\,|x|\right)\right) = +p^3\sin\left(|p|\,|x|\right),$$

for example,

$$p^{7-2}\sin\left(|p|\,|x|\right) = +\frac{d^{7-3}}{dx^{7-3}}\left(p\sin\left(|p|\,|x|\right)\right)$$
$$p^5\sin\left(|p|\,|x|\right) = p^2\frac{d^3}{dx^3}\left(\cos\left(|p|\,|x|\right)\right) = -p^3\frac{d^2}{dx^2}\left(\sin\left(|p|\,|x|\right)\right) = -p^4\frac{d}{dx}\cos\left(|p|\,|x|\right) =$$
$$= +p^5\sin\left(|p|\,|x|\right)$$

then, the expression (31) becomes

$$\begin{aligned} I &= \int e^{-i(1\mp i\varepsilon)\sqrt{p^2+m^2}t}\frac{p^{N-2}}{|x|}dp\;2\sin\left(|p|\,|x|\right) = \\ &= \frac{2}{|x|}\left(\frac{id}{dx}\right)^{N-3}\left[\int e^{-i(1\mp i\varepsilon)\sqrt{p^2+m^2}t}pdp\sin\left(|p|\,|x|\right)\right] = \\ &= 2\left(\frac{id}{dx}\right)^{N-3}\left[\pm\frac{2i}{(2\pi)^2}\lim_{\varepsilon\to 0}\theta\left(\pm t\right)it\left(1\mp i\varepsilon\right)m^2\frac{K_2\left(m\sqrt{x^2-t^2(1\mp i\varepsilon)^2}\right)}{x^2-t^2(1\mp i\varepsilon)^2}\right], \end{aligned} \tag{33}$$

consequently for the case of $(2M + 1, 1)$ dimensions (e.g., even dimensions) we have

$$G^{\pm}(t,x) = \pi 2^N \left(\frac{id}{dx}\right)^{N-3} \left[\pm \frac{2i}{(2\pi)^2} \lim_{\varepsilon\to 0} \theta(\pm t)\, it\,(1\mp i\varepsilon)\, m^2 \frac{K_2\left(m\sqrt{x^2 - t^2(1\mp i\varepsilon)^2}\right)}{x^2 - t^2(1\mp i\varepsilon)^2} \right]. \quad (34)$$

Remark 2. The integrability in even dimensions is due to an underlying symplectic structure ("phase space map," only valid in even number of spacetime dimensions). ■

V. RELATIVISTIC POSITION OPERATOR FROM VELOCITY OPERATOR

A. Velocity operator: definition and action

Canonically the reasonable assumption as our starting point are the following relations:

$$H(x,p) \equiv \sqrt{p^2 + m^2} \quad (35)$$

and

$$\frac{dp}{dt} = -\frac{\partial H(x,p)}{\partial x}, \quad (36)$$

$$\frac{dx}{dt} = \frac{\partial H(x,p)}{\partial p}. \quad (37)$$

Then, the velocity operator is defined from

$$\widehat{u}^{\alpha} = \frac{d\widehat{x}^{\alpha}}{dt} = -i\left[H(x,p), \widehat{x}^{\alpha}\right] \quad (38)$$

$$= \frac{\partial H(x,p)}{\partial p_{\alpha}} = \frac{\delta^{\alpha\beta} p_{\beta}}{\sqrt{p^2 + m^2}}, \quad (39)$$

with the action as

$$(\widehat{u}^{\alpha}\psi)(t,x) = \frac{1}{(2\pi)^3} \int\int \widehat{u}^{\alpha}(\zeta)\, e^{i\zeta\cdot(x-y)} \psi(t,y)\, d^3y\, d^3\zeta, \quad (40)$$

$$(\widehat{u}^{\alpha}\psi)(t,x) = \frac{1}{(2\pi)^3} \int\int \frac{\zeta^{\alpha}}{\sqrt{\zeta^2 + m^2}} e^{i\zeta\cdot(x-y)} \psi(t,y)\, d^3y\, d^3\zeta \quad (41)$$

$$= \frac{1}{(2\pi)^3} \int\int \frac{\zeta^{\alpha}}{\sqrt{\zeta^2 + m^2}} e^{i\zeta\cdot(x-y)} \psi(t,y)\, d^3y\, d^2\zeta.$$

In 1+1 dimensions we have

$$(\widehat{u}^{\alpha}\psi)(t,x) = \frac{1}{2\pi} \int\int \frac{\zeta^{\alpha}}{\sqrt{\zeta^2 + m^2}} e^{i\zeta\cdot(x-y)} \psi(t,y)\, dy\, d\zeta \quad (42)$$

and knowing that

$$\psi(t,y)|_{1+1} = \frac{1}{2\pi} tm \frac{K_1\left(m\sqrt{y^2 - t^2}\right)}{\sqrt{y^2 - t^2}} \quad (43)$$

we insert it in (42), then, the equation to solve becomes

$$(\widehat{u}^{\alpha}\psi)(t,x) = \frac{1}{2\pi} \int\int \frac{\zeta^{\alpha}}{\sqrt{\zeta^2 + m^2}} e^{i\zeta\cdot(x-y)} \frac{1}{2\pi} tm \frac{K_1\left(m\sqrt{y^2 - t^2}\right)}{\sqrt{y^2 - t^2}}\, dy\, d\zeta. \quad (44)$$

Using formula GR 3.365 (2) in the $m \neq 0$ case (the non-massive case will be analyzed separately), the expression to integrate in y is

$$(\widehat{u}^{\alpha}\psi)(t,x) = \frac{1}{(2\pi)^2} \int\int im^2 K_1(m(x-y))\, t \frac{K_1\left(m\sqrt{y^2 - t^2}\right)}{\sqrt{y^2 - t^2}}\, dy, \quad (45)$$

consequently, the action of the velocity operator on the state solution given by the "square root" Hamiltonian is (using GR 7[8])

$$(\widehat{u}^{\alpha}\psi)(t,x) = \frac{1}{(2\pi)^2}\frac{m^2t^2}{\sqrt{2}}K_1\left(m\sqrt{2(x^2-t^2)}\right) \qquad (K_{-\nu} = K_{\nu}). \tag{46}$$

The extension to more dimensions is straightforward.

B. Determination of the relativistic position operator

Theorem 3. *The canonical position operator in p-representation, namely, ∂_p, acting on the convoluted state (23) (equivalent to the initial condition in the operatorial approach) determines (in the case of null eigenvalue) univoquely and simultaneously the action of the velocity operator in the p representation plus the ground state of the physical system under consideration.*

Proof. From the initial condition (23) (the kernel can be straightforwardly extended to any dimension) we have

$$\Psi_0(p) \equiv ae^{-i\left(\sqrt{p'^2+m^2}\beta - p'\alpha\right)}. \tag{47}$$

Operating with $\widehat{x} \to \widehat{\partial}_p$ (e.g.: momentum representation) we can see that

$$\partial_p\Psi_0(p) = -i\left[\frac{p\beta}{\sqrt{p^2+m^2}} - \alpha\right]\Psi_0(p). \tag{48}$$

Then

$$\partial_p\Psi_0(p) = 0 \to \frac{p\beta}{\sqrt{p^2+m^2}}\Psi_0(p) = \alpha\Psi_0(p), \tag{49}$$

we obtain the action of the relativistic velocity operator (40) and (41) analyzed previously. ■

Consequently we can arrive to the following.

Remark 4. Conversely, the respective eigenvalues of the relativistic velocity operator are obtained from the null eigenvalue of the (standard) canonical position operator in the momentum representation.

VI. DISCUSSION

The meaning of the convoluted initial state $\Psi_0(p)$ can be interpreted as follows: the conjugate variable to x^0 (identified with the *physical time*)

$$p^0 = -i\partial^0 \tag{50}$$

is no longer well defined because clearly it must be expressed in terms of the remaining variables. Then, the following identification is immediately performed:

$$\Psi_0(p) \equiv ae^{-i\left(\sqrt{p'^2+m^2}\beta - p'\alpha\right)}, \tag{51}$$

$$= ae^{-i\left(P^0\beta - p^i\alpha_i\right)}, \tag{52}$$

where $P^0 = \sqrt{-\partial^i\partial^i + m^2}$ and $p^i = -i\partial^i$ at $x^0 = 0$, with the consequence that the remaining generators of the Lorentz group can be determined

$$M^{ij} = i\left(x^j\partial^i - x^i\partial^j\right), \tag{53}$$

$$M^{0i} = -\frac{1}{2}\left\{x^i, P^0\right\}, \tag{54}$$

including the fact that x^i and P^0 not commute. The generators on the initial surface $x^0 = 0$ (with conjugate variable p^0) split into *kinematical generators* $M^{ij}(rotations)$, $p^i(momenta)$, and the *dynamical* ones $M^{0i}(boost)$ and $P^0(Hamiltonian)$ that displaces the system away from the initial surface. From the algebraic point of view we can check from the definition of the velocity operator

$$\widehat{u}^\alpha = \frac{d\widehat{x}^\alpha}{dt} = -i\left[H(x,p), \widehat{x}^\alpha\right] = \frac{\partial H(x,p)}{\partial p_\alpha} = \frac{\delta^{\alpha\beta} p_\beta}{\sqrt{p^2+m^2}} \tag{55}$$

$$= -\frac{1}{2} i M^{0i}, \tag{56}$$

that is directly related with the boost generator (notice the interplay between the Poisson and quantum structure). As is easy to interpret, if we have into account the spinorial (Clifford) structure of the double covering of the Lorentz group, namely, the $SL(2\mathbb{C})$, the spin degrees of freedom can be introduced (besides the orbital part)

$$M^{ij} = i\left(x^j\partial^i - x^i\partial^j\right) + \varepsilon^{ijk} S^k, \tag{57}$$

$$M^{0i} = -\frac{1}{2}\left\{x^i, P^0\right\} - \frac{i\varepsilon^{ijk}\partial^j S^k}{P^0 \pm m}, \tag{58}$$

however, if we also add the space time translations: $x^\mu \to x^\mu + a^\mu$, the Poincare group (as semidirect product of the Lorentz group plus the space-time translations) acts on Hilbert states labeled by vectors of the form

$$\left|p^i; m, s, s_3\right\rangle, \tag{59}$$

which are interpreted as physical states (particles) with mass m, spin s, 3-momentum p^i, and magnetic quantum number s_3. The positive or negative energy depends on the sign of β into the exponential of $\Psi_0(p)$. By the way, notice that this fact is connected with the Lagrange multiplier prescription of Dirac[16] where

$$P^0 = p^0 + \lambda\left(p^\mu p_\mu + m^2\right) \tag{60}$$

then, eliminating p^0 through λ (e.g., taking P^0 as Hamiltonian) we have $P^0 = p^0 = \pm\sqrt{p^i p^i + m^2}$ corresponding to positive and negative energy solutions.

The familiar interpretation of the eigenvalues of an observable as the only possible values that can result from measurements of the observable on any state of the system is no longer tenable, because the expectation value of an observable in a particular state and the average of the eigenvalues of the observable weighted by the absolute square of the amplitude of the corresponding eigenstate in the state under question are not equal. A simple but telling example is the case of the operator of the Sakata-Taketani: the velocity operator in the conventional language, has only zero eigenvalues, yet it is an observable in the sense of pseudo-hermiticity and has a non-vanishing expectation value in an arbitrary state. As was pointed out before,[20] this kind of difficulty connected with the appearance of the indefinite metric renders the choice between different possible operators for an observable quantity much more difficult than in the spin 1/2 case.

VII. MP(n) AND THE ALGEBRAIC INTERPRETATION OF THE SQUARE ROOT OPERATOR

Geometrically, in our early work,[1] we take as the starting point the action functional that will describe the world-line of the superparticle (measure on a superspace) as follows:

$$S = \int_{\tau_1}^{\tau_2} d\tau L\left(x, \theta, \overline{\theta}\right) = -m\int_{\tau_1}^{\tau_2} d\tau\sqrt{\overset{\circ}{\omega}_\mu \overset{\circ}{\omega}{}^\mu + \mathbf{a}\dot{\theta}^\alpha\dot{\theta}_\alpha - \mathbf{a}^*\dot{\overline{\theta}}^{\dot{\alpha}}\dot{\overline{\theta}}_{\dot{\alpha}}}, \tag{61}$$

where $\overset{\circ}{\omega}_\mu = \dot{x}_\mu - i(\dot{\theta}\ \sigma_\mu\overline{\theta} - \theta\ \sigma_\mu\dot{\overline{\theta}})$, and the dot indicates derivative with respect to the parameter τ, as usual. The above Lagrangian (we will not give the details here) was constructed considering the line element (e.g., measure, positive square root of the interval) of the non-degenerated supermetric introduced in Ref. 1 $ds^2 = \omega^\mu\omega_\mu + \mathbf{a}\omega^\alpha\omega_\alpha - \mathbf{a}^*\omega^{\dot{\alpha}}\omega_{\dot{\alpha}}$, where the bosonic term and the Majorana bispinor compose a superspace $(1,3|1)$, with coordinates $(t, x^i, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, and where Cartan forms

of supersymmetry group are described by $\omega_\mu = dx_\mu - i(d\theta\sigma_\mu\bar{\theta} - \theta\sigma_\mu d\bar{\theta})$, $\omega^\alpha = d\theta^\alpha$, $\omega^{\dot{\alpha}} = d\theta^{\dot{\alpha}}$ (obeying evident supertranslational invariance).[49] To do this, we take the coordinates $x(\tau)$, $\theta^\alpha(\tau)$, and $\overline{\theta}^{\dot{\alpha}}(\tau)$ depending on the evolution parameter τ. The Hamiltonian in square root form, namely, $\sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^\alpha\Pi_\alpha - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)}|\Psi\rangle = 0$, was constructed defining the supermomenta as usual and, due to the nullification of this Hamiltonian, the Lanczos method for constrained Hamiltonian systems was used.

Consequently, we have shown that there exist an algebraic interpretation of the pseudodifferential operator (square root) in the case of an underlying Mp(n) group structure

$$\sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^\alpha\Pi_\alpha - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)}|\Psi\rangle = 0, \tag{62}$$

$$\left\{\left[m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^\alpha\Pi_\alpha - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)\right]^\alpha_\beta (\Psi L_\alpha)\right\}\Psi^\beta = 0, \tag{63}$$

then, both structures can be identified, e.g., $\sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^\alpha\Pi_\alpha - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)}$ $\leftrightarrow \left[m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^\alpha\Pi_\alpha - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)\right]^\alpha_\beta (\Psi L_\alpha)$ being the state Ψ the square root of a spinor Φ (where the "square root" Hamiltonian acts) such that it can be bilinearly defined as $\Phi = \Psi L_\alpha \Psi$. Our goal in these references was based on the observation that the operability of the pseudodifferential "square root" Hamiltonian can be clearly interpreted if it acts on the square root of the physical states. In the case of the metaplectic group, the square root of a spinor certainly exist [Refs. 44, 14, 13, and 15] making this interpretation (62) and (63) fully consistent from the relativistic and group theoretical viewpoint.

It is interesting to note, that in Ref. 17 the Dirac factorization of the one dimensional relativistic Schrodinger equation was treated introducing the so called quantum simulation of the Dirac equation.[18] This is, in effect, a toy model apparently capable to simulate a genuine quantum relativistic effect, as the zitterbewegung. However, the vector $\underline{\Psi}$ in Ref. 17 is not a spinor and Eq. (37) from Ref. 17 is not the relativistic counterpart of the Pauli equation: there are not spin degrees of freedom and relativistic invariance. The construction given there is only a mathematical artifact in order to mimify the relativistic effects in a sharp contrast with Equation (63) that is fully relativistic and capable of including a complete (super) multiplet (spanning spins from 0 , 1/2, 1, 3/2, 2) of physical states. In the next paragraph, we will describe these states (truly spinorial and relativistic ones) obtained from the algebraic correspondence in order to compare them with their respective results of the quantum simulation of the Dirac equation results presented in Ref. 17.

A. Superspinorial zitterbewegung

Now we will pass to analyze and review the description given in Ref. 29 to see the origin of the quantum relativistic effects as the zitterbewegung. Concerning to the solutions obtained in the "algebro-pseudodifferential" correspondence, we must regard that there are two types of states: the basic (non-observable) ones and observable physical states (see from another point of view the results of Refs. 1, 2, 34, and 29). The basic states are coherent states corresponding to the double covering of the $SL(2C)$[33] or the metaplectic group[1,2,29,34] responsible for projecting the symmetries of the 6 dimensional $Mp(4)$ group space to the 4 dimensional spacetime by means of a bilinear combination of the $Mp(4)$ generators.[35,36]

Regarding previous works,[1,2] the supermultiplet solution for the geometric Lagrangian was

$$g_{ab}(0,\lambda) = \langle\psi_\lambda(t)|\, L_{ab}\, |\psi_\lambda(t)\rangle = e^{-\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2} e^{\xi\varrho(t)} \chi_f \langle\psi_\lambda(0)| \begin{pmatrix} a \\ a^\dagger \end{pmatrix}_{ab} |\psi_\lambda(0)\rangle,$$

where we have written the corresponding indices for the simplest supermetric state solution being L_{ab} the corresponding generators $\in Mp(n)$ in the representation given in Refs. 13 and 44 and χ_f, coming from the odd generators of the big covering group related to the symmetries of the specific

model (will not be treated in this paper, and will be left aside). Considering, for simplicity, the "square" solution for the three compactified dimensions[2] (spin λ fixed, $\xi \equiv -\left(\overline{\xi}^{\dot{\alpha}} - \xi^{\alpha}\right)$). We have obtained schematically for the exponential even fermionic part

$$\varrho(t) \equiv \overset{\circ}{\phi}_{\alpha}\left[\left(\alpha e^{i\omega t/2} \;+\; \beta e^{-i\omega t/2}\right) - \left(\sigma^{0}\right)^{\alpha}_{\dot{\alpha}}\left(\alpha e^{i\omega t/2} - \beta e^{-i\omega t/2}\right)\right] \tag{64}$$

$$+\frac{2i}{\omega}\left[\left(\sigma^{0}\right)_{\alpha}^{\dot{\beta}}\;\overline{Z}_{\dot{\beta}} + \left(\sigma^{0}\right)^{\alpha}_{\dot{\alpha}}\;Z_{\alpha}\right], \tag{65}$$

where $\overset{\circ}{\phi}_{\alpha}, Z_{\alpha}, \overline{Z}_{\dot{\beta}}$ are constant spinors, and α and β are $\mathbb{C}$-numbers (the constant $c_1 \in \mathbb{C}$ due to the obvious physical reasons and the chirality restoration of the superfield solution [1,2,10]).

By consistency, as in the case of the string, two geometric-physical options will be related to the orientability of the superspace trajectory:[12] $\alpha = \pm\beta$. We have take, without lose generality $\alpha = +\beta$ then, exactly, there are two possibilities they are as follows:

(i) the compact case (which was given before in Refs. 2 and 1)

$$\varrho(t) = \begin{pmatrix} \overset{\circ}{\phi}_{\alpha}\cos(\omega t/2) + \dfrac{2}{\omega}Z_{\alpha} \\ -\overset{\circ}{\overline{\phi}}_{\dot{\alpha}}\sin(\omega t/2) - \dfrac{2}{\omega}\overline{Z}_{\dot{\alpha}} \end{pmatrix} \tag{66}$$

(ii) and the non-compact case

$$\varrho(t) = \begin{pmatrix} \overset{\circ}{\phi}\cosh(\omega t/2) + \dfrac{2}{\omega}Z_{\alpha} \\ -\overset{\circ}{\overline{\phi}}_{\dot{\alpha}}\sinh(\omega t/2) - \dfrac{2}{\omega}\overline{Z}_{\dot{\alpha}} \end{pmatrix} \tag{67}$$

obviously (in both cases) represents a *Majorana fermion* where the $\mathbb{C}$ (or $hypercomplex$ wherever the case) symmetry is inside of the constant spinors.

The spinorial even part of the superfield solution in the exponent becomes to

$$\xi\varrho(t) = \theta^{\alpha}\left(\overset{\circ}{\phi}_{\alpha}\cos(\omega t/2) + \frac{2}{\omega}Z_{\alpha}\right) - \overline{\theta}^{\dot{\alpha}}\left(-\overset{\circ}{\overline{\phi}}_{\dot{\alpha}}\sin(\omega t/2) - \frac{2}{\omega}\overline{Z}_{\dot{\alpha}}\right) \tag{68}$$

for the $\mathbb{C}$ (or $hypercomplex$ wherever the case) symmetry. We easily see that in the above expression there appear a type of *zitterbewegung* or continuous oscillation between the chiral and antichiral part of the bispinor $\varrho(t)$. (see, for example, Figures 1–4 in Ref. 29 are snapshots describing the time evolution of the oscillating effect for suitable values of the parameters of the vacuum solution and with an increasing $\omega t \sim t/|a|$, respectively ($\omega_1 < \omega_2 < \omega_3 < \cdots$)).

Remark 5. the physical meaning of such an oscillation (zitterbewegung) is simply an underlying natural supersymmetric effect because there exists a kind of duality between supersymmetrical and relativistic effects, pointed out previously in Ref. 4.

VIII. PHYSICAL EXAMPLES

A. Rogue waves

As is more or less understandable from their scientific observation at the Draupner oil platform in the North Sea,[5] rogue waves (sometimes described as monster waves, freak waves, or giant waves) appear with an amplitude extremely larger with respect to the amplitude of the surrounding wave crests.[6] Because the conditions that cause the enormous growth of rogue waves are still not well known, they have become a subject of intense research after their experimental realization and simulation in various physical systems having an underlying nonlinear character, namely, optical fibers[7,8] and plasmas.[9,10]

The general form to attack the problem in many of these contexts is to introduce different variants of the nonlinear Schrodinger equation (NLSE) due to the modulation of its present instability.[11]

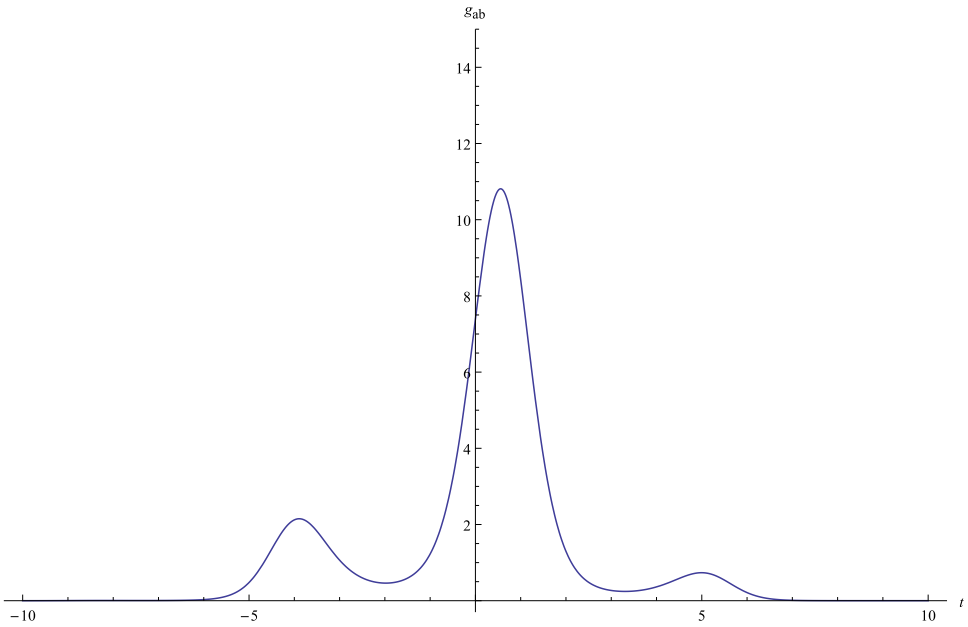

FIG. 1. Snapshot describing the evolution of the oscillating effect in complex time.

This kind of modulation can be effectively implemented at laboratory level in the Bose-Einstein condensation (BEC) scenarios where the Feshbach resonance technique[19] allows to control the dynamics of matter rogue waves by means of the feasibility of tuning interatomic interactions. Also, in the same context of BEC, the quasi-one dimensional Gross-Pitaevskii (GP) as the NLSE with trapping potential is usually utilized.

The interesting point is that in recent years the study of spinor condensates has been an important issue experimentally and theoretically speaking.[21] As is well known, the dynamics of the spinor condensate is described within the mean-field approximation by the multicomponent GP-equations containing nontrivial nonlinear terms mimifiying the SU(2) symmetry of the spins. Alkali-metal atoms are usually represented in a such manner. Now, we can see that rogue waves can be obtained by the free Schrodinger equation solution in the algebro-pseudodifferential framework given above. Controlling the parameters of the solution given by expressions (64)-(68), the expected waves are obtained when the time variable has complex coefficients (e.g., $\rightarrow it$). It is clear that it is a kind of "dispersion in time" that is responsible for the "exploded" wave dynamics. 3-dimensional Figures 5–7 show the wave behavior with an increasing imaginary coefficient in the time variable, respectively. Figures 1–4 show snapshots describing the evolution of the oscillating effect in complex time.

B. The Nambu-Goto action and the microcanonical propagator

Here we will make some comments about the pseudodifferential operators and physical systems with finite energy (e.g., microcanonical ensemble) in connection with the quantum field theoretical (QFT) viewpoint. Regarding our previous reference (see full details in Ref. 43 and connected with a string-high energy framework see Ref. 45) the propagator for black-hole/string/particle was constructed knowing that the Nambu-Goto action is invariant under the reparametrizations. Using the "Born-Infeld" choice for the dynamical variables,[30] we obtain the action in the form

$$S = -\frac{\kappa}{\alpha'}\int_{\tau 1}^{\tau 2} \dot{x}_0 d\sigma\, d\tau \sqrt{\left[1-(\partial_0 x_b)^2\right]\left[1+(\partial_1 x_a)^2\right]}, \qquad \left(a,b=2,3;\ \partial_1 x_a = \varepsilon_{1a}{}^{0b}\partial_0 x_b\right). \tag{69}$$

Therefore, the invariance with respect to the invariance of the coordinate evolution parameter means that one of the dynamic variables of the theory ($x_0(\tau)$ in this case) becomes the observed time with the corresponding non-zero Hamiltonian $H_{BI} = \Pi_a \dot{x}^a - L = \sqrt{\alpha^2 - \Pi_b \Pi^b}$, where: $\Pi^b =$

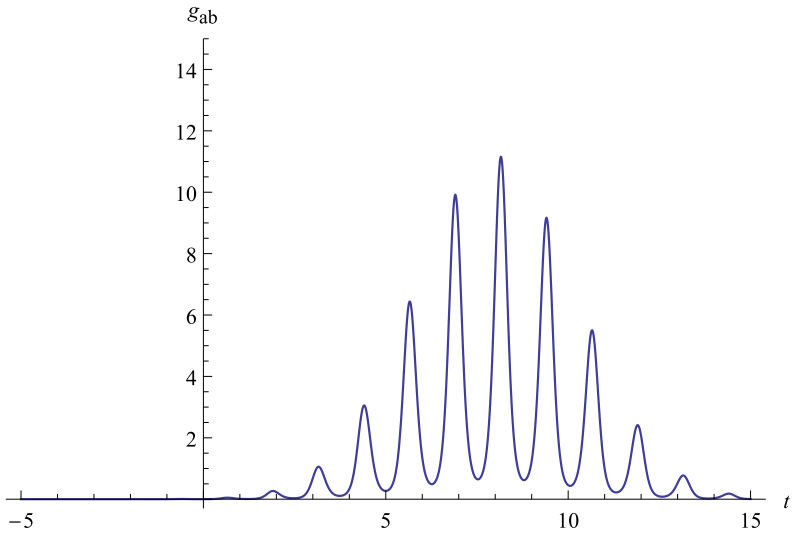

FIG. 2. Snapshot describing the evolution of the oscillating effect in complex time.

$\frac{\partial L}{\partial(\partial_0 x_b)}, \alpha \equiv \frac{\kappa\sqrt{1+(\partial_1 x_a)^2}}{\alpha'}$. From the simplest path-integral formalism, using quantum field theoretical arguments and introducing the integral representation for a pseudodifferential operator[31] (based on semigroup construction),

$$\int \left(t^2+u^2\right)^{-\lambda} e^{itx} dt = \frac{2\pi^{1/2}}{\Gamma(\lambda)}\left(\frac{|x|}{2u}\right)^{\lambda-1/2} K_{\lambda-1/2}(u|x|), \tag{70}$$

where $K_\nu(x)$ is the MacDonald's function, we obtain[45,43] the following microcanonical propagator:

$$D_E(t,\overline{x}) = \frac{\delta(E)}{\omega^2-k^2-m^2+i\varepsilon} -$$
$$-8\pi i\alpha\delta\left(\omega^2-k^2-m^2\right)\sum_{l=1}^{\infty}\frac{K_{-1}(\alpha|l\omega_k-E|)}{E^2}\frac{\Omega(E-l\omega_k)}{\Omega(E)}\theta(E-l\omega_k), \tag{71}$$

where $\theta(x)$ is the usual step function. The first term in the microcanonical propagator is the usual (non-termal) Feynman propagator, the second one is the new microcanonical statistical part. The

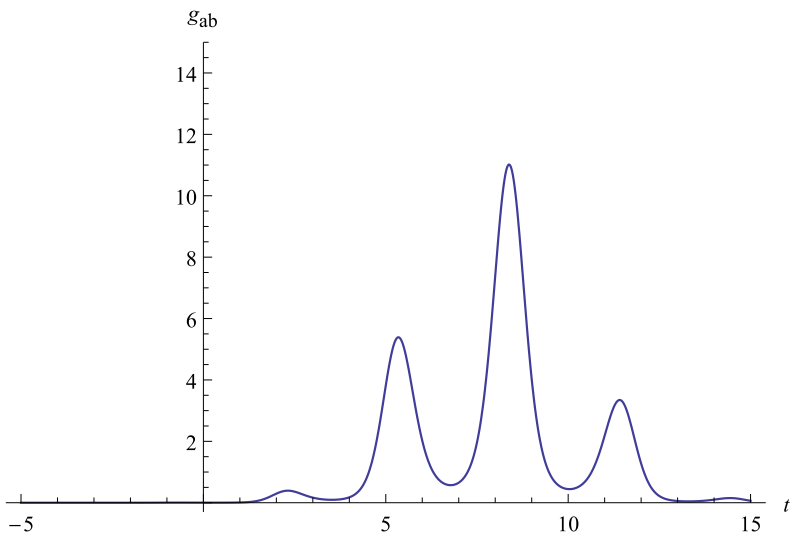

FIG. 3. Snapshot describing the evolution of the oscillating effect in complex time.

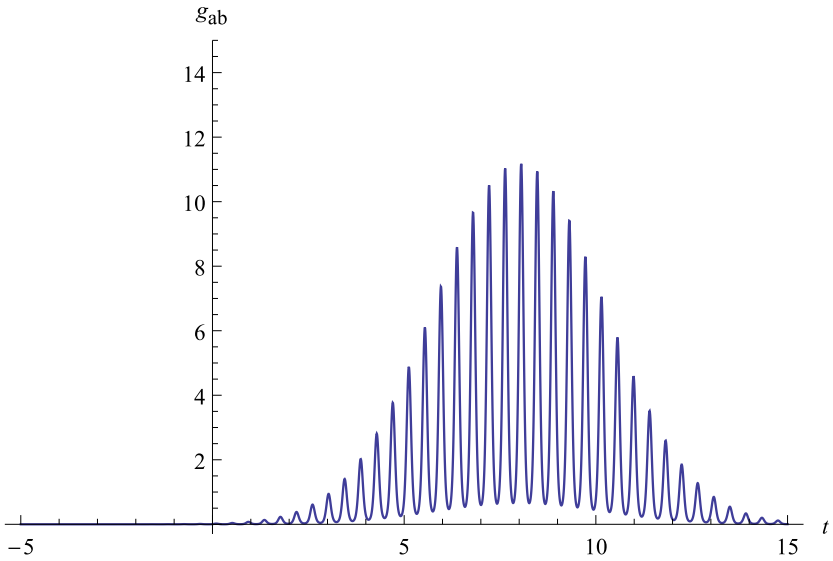

FIG. 4. Snapshot describing the evolution of the oscillating effect in complex time.

correct description of the full N-extended body system is obtained explicitly expanding the Mac Donald's function K_{-1} in the second term of the free microcanonical propagator[45] leading to a nonlocal and nonlinear generalization of the well known (string-theoretical) Veneziano amplitude.[32] This observation leads us to highlight the next.

Remark 6. Pseudodifferential operators in QFT give rise to propagators in which a string-like type of structure emerges (gamma type string-amplitude) contributing to their statistical relation between temporal and normal ordering of the field operators.

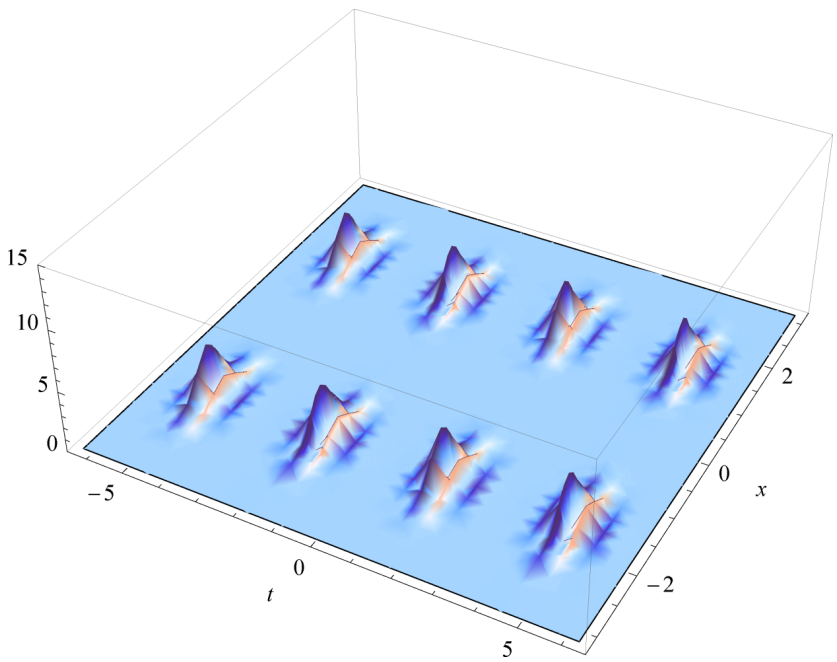

FIG. 5. Wave behavior with an increasing imaginary coefficient in the time variable.

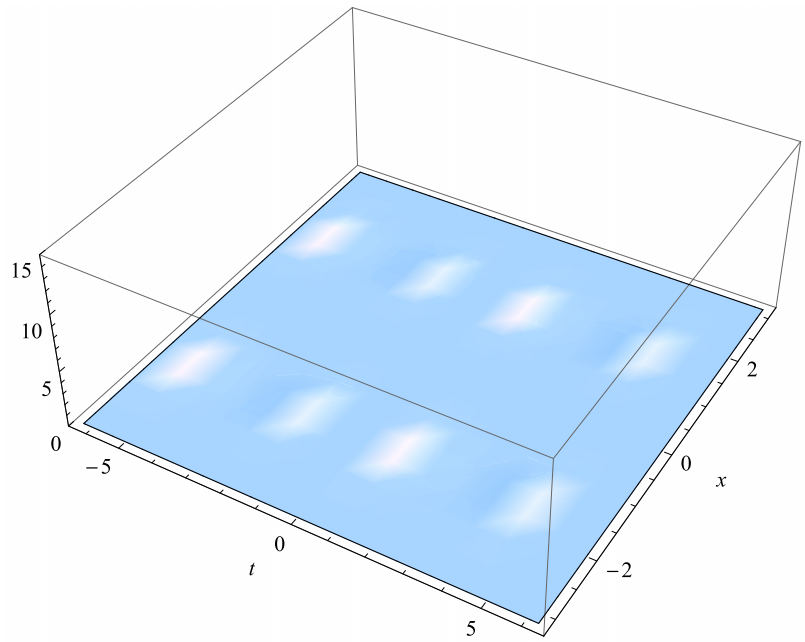

FIG. 6. Wave behavior with an increasing imaginary coefficient in the time variable.

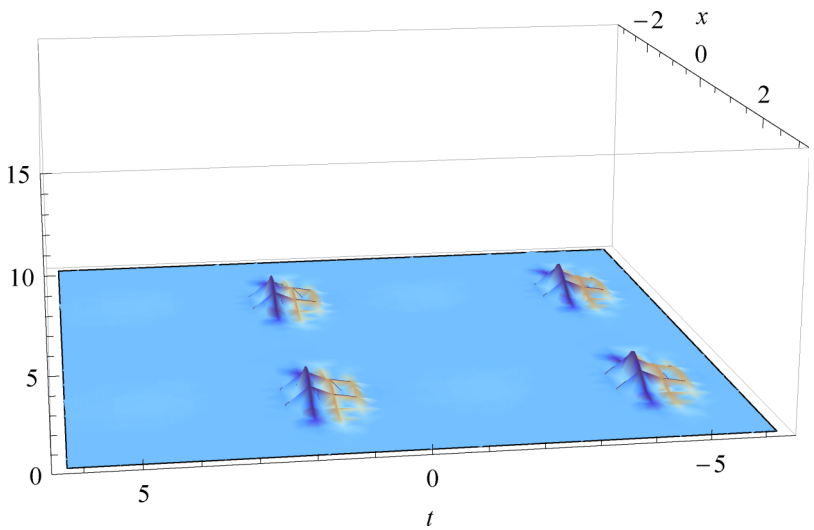

FIG. 7. Wave behavior with an increasing imaginary coefficient in the time variable.

C. Warped gravities, randall sundrum scenarios, and the square root

The last example coming from our Ref. 2 shows the consistency of this interpretation. The motivation to introduce pseudodifferential operators was to find the consistent solution to the hierarchy problem[39] and, due to the lack of formal "first principles" explanations, to the field theoretical localization mechanisms for scalar and fermions[40] as well as for gauge bosons.[41] Some points coming from the analysis of these previous works[2] must be highlighted.

The remarkable property of the full solution involving beside the expressions (63)-(67), the bosonic part namely, $g_{ab}(t) = e^{-\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + c_1 t + c_2 + \xi \varrho(t)} g_{ab}(0)$[2], is that the physical state $g_{ab}(x)$ is localized in a particular position of the space-time: the supermetric $\mathbb{C}$ coefficients $\mathbf{a}\,(\mathbf{a}^*)$ play the important role of localizing the fields in the bosonic part of the superspace in a similar and suggestive

form as the well known "warp factors" in multidimensional gravity[42] for a positive (or negative) tension brane. This Gaussian type solution is a very well defined physical state in a Hilbert space[38,37] from a mathematical point of view, contrarily to the usual case $u(y) = ce^{-H|y|}$ given in Ref. 42 that, although were possible to find a manner to include it in any Hilbert space, is strongly needed to take special mathematical and physical particular assumptions whose meaning is obscure. For a more complete picture the comparison with the case of 5-dimensional gravity plus cosmological constant[42] is clearly given with full details in the table of reference.[50]

IX. CONCLUDING REMARKS

In this work, we have made a development and analysis of the problem generated by the "square root" operator. Through this paper we logically emphasize the non-locality and the relation with the Green function approach.

We show that there exists a close relation between the number of spacetime dimensions and the order of the cylinder functions (MacDonald's function in our case) having the case of even number of spacetime dimensions an exact integrability.

The self-reproducing property of the MacDonald's function, as the main ingredient of the Green kernel, makes possible the straightforward relation with the coherent and squeezed states of the non-compact groups (e.g., $SU(1,1)$) and the corresponding double coverings as is the metaplectic typical case.[44] In this sense, we have demonstrated here clearly, through the comparison with Ref. 46, the relation with the non-Hermitian time operator and, looking at Ref. 47, the relation with the time-energy coherent states. In this manner we have shown specifically the form of the "overlap" integrals and the physical operators of the "observables": velocity and the phase space structure. The conclusion that is immediately obtained from this last point is that the coherent states structure is related with the time dispersion of the non-local square root Hamiltonian.[48]

The integrability for several dimensions is achieved in the case of even dimensions due to the symplectic (phase space) underlying structure. All remaining aspects concerning these issues were clearly treated through the first part of the paper by means of respective theorems, proofs, and remarks.

The algebraic connection with the pseudodifferential description, described from our previous works and formally proposed by us here, allows the correct interpretation of the square root treated as operator that has been exemplified by three physical cases (namely, rogue waves, warped gravities, and the Nambu-Goto action and the microcanonical propagator). The relativistic wave equations for any spin are also described by this algebraic interpretation establishing a bridge with the pseudodifferential and semigroup approaches.

An important new result, in the context of the algebraic approach, that we have found before is that there exists an oscillatory fermionic effect in the B_0 part of the supermultiplet as a Zitterwebegung, but between the chiral and antichiral components of this Majorana bispinor. This effect is (see Equation (67)) fully relativistic and capable of including a complete (super) multiplet of physical states in a sharp contrast with Ref. 17 where the Dirac factorization of the one dimensional relativistic Schrodinger equation was treated introducing the so called quantum simulation of the Dirac equation,[18] where the vector $\underline{\Psi}$ in Ref. 17 is not a spinor and Eq. (37) from Ref. 17 is not the relativistic counterpart of the Pauli equation: there are no spin degrees of freedom and relativistic invariance.

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[48] It is useful to know that we can adimensionalize our problem introducing $\tau \equiv \frac{ct}{\lambda_c}, \eta \equiv \frac{\hbar x}{mc}\left(\lambda_c = \frac{h}{mc}\right)$ then, the Schrodinger equation becomes $\partial_\tau \psi(\tau,\eta) = \sqrt{1-\partial_\eta^2}\psi(\tau,\eta)$. In the other case, we take through this paper: $\hbar = 1 = c$.
[49] As we have extended our manifold to include fermionic coordinates, it is natural to extend also the concept of trajectory of point particle to the superspace.
[50] The extended superspace solution in the case contains all the 4-dimensional coordinates: $x \equiv (t, \overline{x})$, $c_1' x \equiv c_{1\mu}' x^\mu$, and c_2' scalar.